Concepts and multilattices and the distributivity of the Dedekind-MacNeille completion*

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Abstract. The Dedekind-MacNeille completion of a poset P can be seen as the least complete lattice containing P. In this work, we analyze some results concerning the use of this completion within the framework of Formal Concept Analysis, notably the distributivity of the Dedekind-MacNeille completion and the construction of the poset of concepts associated with a Galois connection between posets.

1 Introduction

The Dedekind-MacNeille completion of a partially ordered set P was introduced by H.M. MacNeille in [9] as a generalization of Dedekind's method for constructing the field of the real numbers from the rational numbers. In a few words, one can say that the Dedekind-MacNeille completion of a poset $(P, \leq)$ is the smallest complete lattice that contains P.

This construction has already played a role in the research topic of formal concept analysis in which, for instance, the concept lattice corresponding to the general ordinal scale associated to a poset is precisely the Dedekind-MacNeille completion of P, see [7]. The problem of actually constructing the completion of a finite poset is very interesting from a practical standpoint, and it is not surprising that several researchers have devised algorithms for constructing it.

On the other hand, multilattices are structures in which the restrictions imposed on a (complete) lattice, namely, the "existence of *least* upper (resp. *greatest* lower) bounds" is weakened to "existence of *minimal* upper (resp. *maximal* lower) bounds". Although introduced in a theoretical framework more than fifty years ago, they are being used as practical tools to handle uncertain information [3, 10]. Specifically, they can be used as structures capable of describing certain aspects of uncertainty and reasoning with incomplete information.

Precisely, it is in this respect where one finds the link between multilattices and Formal Concept Analysis (FCA); specifically, related to the many approaches that can be found in papers aimed at generalizing FCA in order to

* Partially supported by the Spanish Ministry of Science through project TIN12-39353-C04 and the network TIN14-56381-REDT, and the ESF Fund CZ.1.07/2.3.00/30.0041.

deal with uncertainty, imprecise data, or incomplete information, which have provided different abstract frameworks [1, 2, 8, 11, 13], ranging from residuated lattices, to non-commutative conjunctors, and to multi-adjoint lattices. Non-commutativity enables passing from adjoint pairs (generalization of conjunction and implication in a residuated lattice) to adjoint triples [4]. Adjoint triples on lattices have proven to be a useful tool when working in fuzzy formal concept analysis. Furthermore, in [12] it was shown that they can play an important role as well within the framework of multilattices, especially in order to form the Galois connections needed to build concepts in a multilattice-based framework.

This paper studies a extension of the usual theory of FCA, in that we seemingly assume the most general framework for the corresponding constructions. Firstly, we aim at showing that the Dedekind-MacNeille completion behaves adequately with respect to the FCA construction of the concepts, in that the completion of the concept poset coincides with the concept lattice of the corresponding completions of the initial posets.

2 Preliminaries

In this section we recall the preliminary definitions of multilattices, Formal Concept Analysis, and Dedekind-MacNeille completion.

2.1 Multilattices

Definition 1. *A* complete lattice *is a poset* $(L, \leq)$ *where every subset of* L *has supremum and infimum.*

When the existence of supremum (infimum) element is replaced by the existence of minimal (maximal) elements of the upper (lower) bounds of a subset, the notion of multilattice arises. In order to formalize this definition, the following notions are needed.

Definition 2. *Let* $(P, \leq)$ *be a poset and* $K \subseteq P$*, we say that:*

- *K is called a* chain *if for every two elements* $x, y \in K$ *we have that either* $x \leq y$ *or* $y \leq x$*.*
- *K is called* antichain *if none of its elements are comparable, i.e., for every different* $x, y \in K$ *we have both* $x \nleq y$ *and* $y \nleq x$*.*

Definition 3. *A poset* $(P, \leq)$ *is called* coherent *if every chain has supremum and infimum.*

The definition of a multilattice is given below.

Definition 4. *A* complete multilattice *is a coherent poset* $(M, \leq)$ *such that for each subset* X *the set of upper (resp. lower) bounds of* X *has minimal (resp. maximal) elements.*

Each minimal (resp. maximal) element of the upper (resp. lower) bounds of a subset is called multisupremum *(resp.* multinfimum*). The set of all multisuprema, resp. multinfima, of* X *will be denoted by* $\operatorname{msup}(X)$*, resp.* $\operatorname{minf}(X)$*.*

Remark 1. Note that, by definition, the set $\mathrm{msup}(X)$, resp. $\mathrm{minf}(X)$, is never empty. Particularly, every complete multilattice has a bottom and a top element. Moreover, note that the two sets $\mathrm{msup}(X)$ and $\mathrm{minf}(X)$ are always antichains.

Proposition 1 ([11]). *Given a complete multilattice $(M, \leq)$, every upper (resp. lower) bound of a subset $X \subseteq M$ is greater (resp. smaller) than at least one multisupremum (resp. multinfimum) of X.*

Although the following remark can be straightforwardly obtained, we prefer to formally state it since it will be used later.

Remark 2. Given $X \subseteq M$, if $\mathrm{minf}(X) \cap X \neq \varnothing$, then X has a minimum.

2.2 Closure operators and closure systems

As the concepts (that is, the basic constructions in FCA) are closed elements under certain constructions, we give here the preliminary notions needed in relation to closure operators and closure systems.

Definition 5. *Given a poset $(P, \leq)$, a* closure operator *on P is a mapping $c\colon P \to P$ which is monotone, inflationary and idempotent. Specifically, this means the following conditions for all $x, y \in P$*

1. $x \leq y$ implies $c(x) \leq c(y)$ 2. $x \leq c(x)$ 3. $c(x) = c(c(x))$

Let L be a complete lattice. A subset $S \subseteq L$ is a closure system *if for all $X \subseteq S$ we have that $\inf(X) \in S$.*

In this case, every closure operator gives rise to a closure system and vice versa, as the following proposition shows.

Proposition 2. *Let c be a closure operator on a complete lattice $(L, \sqcap, \sqcup)$. Then the family $S_c = \{x \in L \mid c(x) = x\}$ of closed elements of L is a closure system, and forms a complete lattice when ordered by inclusion, in which for any $X \subseteq S_c$ the supremum and infimum are defined by*

$$\bigwedge X = \bigsqcap X \qquad \bigvee X = c(\bigsqcup X).$$

Conversely, given a closure system S in L, then $c_S(x) = \bigsqcap\{y \in S \mid x \leq y\}$ defines a closure operator c_S on L.

2.3 Galois connections and Formal Concept Analysis

The notion of Galois connection [5], which we recall here, will play as well an important role hereafter.

Definition 6. *Let ${}^{\downarrow}\colon P \to Q$ and ${}^{\uparrow}\colon Q \to P$ be two maps between the posets $(P, \leq)$ and $(Q, \leq)$. The pair $({}^{\uparrow}, {}^{\downarrow})$ is called a* Galois connection *if:*

– $p_1 \leq p_2$ *implies* $p_2{}^{\downarrow} \leq p_1{}^{\downarrow}$*, for every* $p_1, p_2 \in P$*;*
– $q_1 \leq q_2$ *implies* $q_2{}^{\uparrow} \leq q_1{}^{\uparrow}$*, for every* $q_1, q_2 \in Q$*;*
– $p \leq p^{\downarrow\uparrow}$ *and* $q \leq q^{\uparrow\downarrow}$*, for all* $p \in P$ *and* $q \in Q$*.*

An interesting property of a Galois connection $({}^{\uparrow}, {}^{\downarrow})$ is that ${}^{\downarrow} = {}^{\downarrow\uparrow\downarrow}$ and ${}^{\uparrow} = {}^{\uparrow\downarrow\uparrow}$, where the chain of arrows means their composition.

Once we have a Galois connection, we can focus on the pairs of elements (p, q) which are the image of each other by the application of the corresponding arrow. These pairs can be seen as fixed points of the Galois connection, and are usually called *concepts*. We formalize this notion as follows:

Definition 7. *A pair* (p, q) *is said to be a* concept *of the Galois connection* $({}^{\uparrow}, {}^{\downarrow})$ *if* $p^{\downarrow} = q$ *and* $q^{\uparrow} = p$*.*

The set of concepts can be ordered by defining $(p_1, q_1) \leq (p_2, q_2)$ if and only if $p_1 \leq p_2$ (or equivalently $q_2 \leq q_1$). The resulting poset will be denoted $\mathsf{CP}(P, Q, {}^{\uparrow}, {}^{\downarrow})$. In the case that P and Q are lattices, the following result holds:

Theorem 1 (See [7]). *Let* $(L_1, \leq_1)$ *and* $(L_2, \leq_2)$ *be two complete lattices and* $({}^{\uparrow}, {}^{\downarrow})$ *a Galois connection between them, then we have that* $\mathsf{CP}(L_1, L_2, {}^{\uparrow}, {}^{\downarrow})$ *is a complete lattice, and the constructions of infima and suprema are given below:*

$$\bigwedge_{i \in I} (x_i, y_i) = \left(\bigwedge_{i \in I} x_i, (\bigvee_{i \in I} y_i)^{\uparrow\downarrow} \right) \qquad \bigvee_{i \in I} (x_i, y_i) = \left((\bigvee_{i \in I} x_i)^{\downarrow\uparrow}, \bigwedge_{i \in I} y_i \right)$$

In this case, we will stress the fact that the set of concepts is a lattice by writing $\mathsf{CL}(L_1, L_2, {}^{\uparrow}, {}^{\downarrow})$.

The following definitions introduce the notion of supremum-dense (resp. infimum-dense) subset, and dual isomorphism, which will be useful later in relation to the basic theorem of FCA for multilattices.

Definition 8. *Let* $(L, \leq)$ *be a lattice and let* $Q \subseteq L$*, we say that the subset* Q *is* supremum-dense *in* L *if for every element* $a \in L$ *there is a subset* $A \subseteq Q$ *such that* a *is the supremum of* A*. The dual of supremum-dense is* infimum-dense*.*

Definition 9. *Let* $(P, \leq)$ *and* $(Q, \leq)$ *be two posets and* φ *a mapping from* P *onto* Q *such that* $x \leq y$ *in* P *if and only if* $\varphi(y) \leq \varphi(x)$ *in* Q*. Then, the mapping* φ *is called* dual isomorphism*.*

2.4 Adjoint triples and Formal Concept Analysis

Finally, we will recall some extensions of notions about formal concept analysis based on the so-called adjoint triples, which can be seen as operators that arise as a generalization of a triangular norm and its residuated implication. These operators will be considered later in Section 5.

Definition 10. *Let* $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ *be posets and consider mappings* $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$, *then* $(\&, \swarrow, \nwarrow)$ *is said to be an* adjoint triple *with respect to* P_1, P_2, P_3, *if* $\&$, $\swarrow, \nwarrow$ *satisfy the* adjoint property*: For all* $x \in P_1$, $y \in P_2$, $z \in P_3$

$$x \leq_1 z \swarrow y \quad \textit{iff} \quad x \,\&\, y \leq_3 z \quad \textit{iff} \quad y \leq_2 z \nwarrow x$$

It is worth to recall that the conjunctor of an adjoint triple was called *biresiduated mapping* in [15].

Definition 11. *A* frame $\mathcal{L}$ *is a tuple* $(L_1, L_2, P, \leq_1, \leq_2, \leq, \&, \swarrow, \nwarrow)$ *where* $(L_1, \leq_1)$ *and* $(L_2, \leq_2)$ *are complete lattices,* $(P, \leq)$ *is a poset and,* $(\&, \swarrow, \nwarrow)$ *is an adjoint triple with respect to* L_1, L_2, P*. These frames are denoted as* $(L_1, L_2, P, \&)$*.*

Given a frame, a *context* is a tuple consisting of sets of objects, attributes and a fuzzy relation among them. Formally,

Definition 12. *Let* $(L_1, L_2, P, \&)$ *be a frame, a* context *is a tuple* (A, B, R) *such that* A *and* B *are non-empty sets (interpreted as attributes and objects, respectively) and* R *is a* P*-fuzzy relation* $R\colon A \times B \to P$*.*

L_1^A and L_2^B denote the set of fuzzy subsets $f\colon A \to L_1$, $g\colon B \to L_2$, respectively. From the partial orders in $(L_1, \leq_1)$ and $(L_2, \leq_2)$, a pointwise partial order can be considered which provides L_1^A and L_2^B with the structure of complete lattice. Abusing notation, $(L_1^A, \leq_1)$ and $(L_2^B, \leq_2)$ are complete lattices where $\leq_1$ and $\leq_2$ are defined pointwise.

Given a fixed frame and a context for that frame, the concept-forming operators ${}^\uparrow\colon L_2^B \longrightarrow L_1^A$ and ${}^\downarrow\colon L_1^A \longrightarrow L_2^B$ are defined, for all $g \in L_2^B$, $f \in L_1^A$ and $a \in A$, $b \in B$, as

$$g^\uparrow(a) = \inf\{R(a, b) \swarrow g(b) \mid b \in B\} \tag{1}$$

$$f^\downarrow(b) = \inf\{R(a, b) \nwarrow f(a) \mid a \in A\} \tag{2}$$

These two arrows form a Galois connection [11]. Therefore, a *fuzzy concept* is a pair $\langle g, f\rangle$ satisfying that $g \in L_2^B$, $f \in L_1^A$ and that $g^\uparrow = f$ and $f^\downarrow = g$; with $({}^\uparrow, {}^\downarrow)$ being the Galois connection defined above.

Definition 13. *The* fuzzy concept lattice *associated with a fuzzy frame* $(L_1, L_2, P, \&)$ *and a context* (A, B, R) *is the set*

$$\mathfrak{B}(A, B, R) = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \textit{ and } g^\uparrow = f, f^\downarrow = g\}$$

in which the ordering is defined by $\langle g_1, f_1\rangle \leq \langle g_2, f_2\rangle$ *if and only if* $g_1 \leq_2 g_2$ *(equivalently* $f_2 \leq_1 f_1$*).*

Notice that $\mathfrak{B}(A, B, R)$ coincides with $\mathsf{CL}(L_1^A, L_2^B, {}^\uparrow, {}^\downarrow)$ and, by Theorem 1, it is a complete lattice.

3 Closure systems in multilattices

This section introduces the definition of closure system in a multilattice, several properties are proved and, finally, the characterization in terms of a closure operator is given. A complete multilattice $(M, \leq)$ will be fixed in all this section,.

The first definition is clearly a natural generalization of a closure system.

Definition 14. *A set $S \subseteq M$ is a* closure system *in M, if for all*[5] *$X \subseteq S$ $\operatorname{minf}(X) \subseteq S$ holds.*

The following results relate closure operators to closure systems on a multilattice. The first one states that the set of fixed points of a closure operator gives rise to a closure system.

Lemma 1. *Let c be a closure operator on M, then the set of fixed points $S_c = \{x \in M \mid c(x) = x\}$ forms a closure system in M.*

The following technical lemma will be fundamental in order to define a closure operator from a closure system.

Lemma 2. *Given a closure system $S \subseteq M$ and $y \in M$, then the set $\{x \in S \mid y \leq x\}$ has a minimum.*

Consequently, given a closure system $S \subseteq M$, the mapping $c_S \colon M \to M$, defined by $c_S(y) = \min\{x \in S \mid y \leq x\}$, is a closure operator on M.

The previous results provide the generalization of the well-known relationship between closure systems and closure operators.

Theorem 2. *Each closure operator on M induces a closure system in M. Conversely, any closure system determines a closure operator.*

Proposition 3. *The closure operator induced by a closure system S_c is c itself, similarly, the closure system induced by the closure operator c_S is S. That is,*

$$c_{S_c} = c \quad \textit{and} \quad S_{c_S} = S$$

Proof. The equality $c_{S_c}(y) = \min\{x \in S_c \mid y \leq x\} = c(y)$ holds, since the closure of y, $c(y)$, is the smallest closed element greater than y.

On the other hand, $S_{c_S} = S$ follows from the fact that $y \in S$ if and only if $c_S(y) = \min\{x \in S \mid y \leq x\} = y$. □

The next result recalls the relation between Galois connections and closure systems in multilattices.

Proposition 4. *Any Galois connection between complete multilattices induces dually isomorphic closure systems. Conversely, each pair of dually isomorphic closure systems S_1 and S_2 in complete multilattices M_1 and M_2 determines a Galois connection between S_1 and S_2.*

[5] Note that the subset X can be empty.

4 Dedekind-MacNeille completion on multilattices

After recalling the notion of Dedekind-MacNeille completion on posets, this section introduces two technical results which will be used later. The following definition presents the two operators used in the completion of an ordered set P.

Definition 15. *Let* $(P, \leq)$ *be a poset and* $A \subseteq P$*, the* "upper" *set and the* "lower" *set of* A *are respectively defined by*

$$A^u = \{x \in P \mid a \leq x, \textit{for all}\, a \in A\} \;\; \textit{and} \;\; A^l = \{x \in P \mid x \leq a, \textit{for all}\, a \in A\}$$

The mappings ^{u} and ^{l} on the powerset of the poset P form a Galois connection. Hence, the following properties hold, for all $A, B \subseteq P$,

$$A \subseteq A^{ul} \text{ and } A \subseteq A^{lu} \tag{3}$$

$$\text{if } A \subseteq B \text{ then } B^u \subseteq A^u \text{ and } B^l \subseteq A^l \tag{4}$$

$$A^u = A^{ulu} \text{ and } A^l = A^{lul} \tag{5}$$

$$\bigcap_{i \in I} (A_i)^u = \Big(\bigcup_{i \in I} A_i\Big)^u, \text{ where } A_i \subseteq P, \text{ for all } i \in I \tag{6}$$

$$\bigcap_{i \in I} (A_i)^l = \Big(\bigcup_{i \in I} A_i\Big)^l, \text{ where } A_i \subseteq P, \text{ for all } i \in I \tag{7}$$

Considering the operators ^{u} and ^{l}, the Dedekind-MacNeille completion of a poset $(P, \leq)$ is defined as follows:

Definition 16 ([5]). *Let* $(P, \leq)$ *be a poset. The* Dedekind-MacNeille completion *of* P *is the set* $\mathsf{DM}(P) = \{A^{ul} \mid A \subseteq P\}$*, which forms a complete lattice with respect to the inclusion ordering.*

It is worth to note that $\mathsf{DM}(P)$ forms a closure system in the powerset of P; consequently, infimum coincides with the intersection and supremum is the closure of the union.

The following theorem characterizes the Dedekind-MacNeille completion of a poset $(P, \leq)$.

Theorem 3 ([5]). *Let* $(P, \leq)$ *be an ordered set and let* $\iota \colon P \hookrightarrow \mathsf{DM}(P)$ *be the order-embedding of* P *into its Dedekind-MacNeille completion given by* $\iota(x) = x^l$*.*

(i) $\iota(P)$ *is both supremum-dense and infimum-dense in* $\mathsf{DM}(P)$*.*
(ii) Let $(L, \leq)$ *be a complete lattice and assume that* P *is a subset of* L *which is both supremum-dense and infimum-dense in* L*. Then* $L \cong \mathsf{DM}(P)$ *via an order-isomorphism which is an extension of* ι*.*

As a result, given a poset $(P, \leq)$, the mapping $\iota \colon P \hookrightarrow \mathsf{DM}(P)$ above is an order-embedding of P into $\mathsf{DM}(P)$.

Another technical result, which will be useful later, is the following:

Proposition 5 ([14]). *For all $X \subseteq P$ the following*[6] *equalities hold in* $\mathsf{DM}(P)$*:*

$$\bigwedge_{x\in X} x^l = X^l \qquad \bigvee_{x\in X} x^l = X^{ul}$$

The following proposition introduces some useful equalities in the case that our underlying poset is indeed a multilattice.

Proposition 6. *For every $X \subseteq M$, the following equalities are satisfied:*

$$X^l = \bigcup_{y\in \mathrm{minf}(X)} y^l \quad \textit{and} \quad X^u = \bigcup_{y\in \mathrm{msup}(X)} y^u$$

Proof. We will prove just the first equality, the second one is similar.

First of all, we will prove that $X^l \subseteq \bigcup_{y\in \mathrm{minf}(X)} y^l$. By definition of multilattice we have that M is a coherent poset, then for all $x \in X^l$ there exists $y \in \mathrm{minf}(X)$, such that $x \leq y$. From the last inequality we obtain that $x \in y^l$ and, as a consequence, $x \in \bigcup_{y\in \mathrm{minf}(X)} y^l$. Therefore, we can conclude that $X^l \subseteq \bigcup_{y\in \mathrm{minf}(X)} y^l$.

It remains to prove that $\bigcup_{y\in \mathrm{minf}(X)} y^l \subseteq X^l$. For that purpose, we will consider $z \in \bigcup_{y\in \mathrm{minf}(X)} y^l$, then $z \in y^l$ for some $y \in \mathrm{minf}(X)$, from which the following inequalities hold $z \leq y \leq x$ for all $x \in X$. Finally, we can state that $z \in X^l$ and, therefore $\bigcup_{y\in \mathrm{minf}(X)} y^l \subseteq X^l$. □

As we know that all the elements in the Dedekind-MacNeille completion of P can be expressed as infima or suprema of elements of P, the following lemma describes how the elements in the completion of a multilattice M can be expressed in terms of elements in M.

Lemma 3. *Let $(M, \leq)$ be a complete multilattice, then for all $X \subseteq M$ the following equalities in* $\mathsf{DM}(M)$ *hold:*

$$\bigwedge_{x\in X} x^l = \bigvee_{y\in \mathrm{minf}(X)} \iota(y) \qquad \bigvee_{x\in X} x^l = \bigwedge_{y\in \mathrm{msup}(X)} \iota(y)$$

Proof. Given $X \subseteq M$, by Proposition 5 we have that $\bigwedge_{x\in X} x^l = X^l$. Then, the following chain of equalities holds

$$\bigvee_{y\in \mathrm{minf}(X)} \iota(y) \stackrel{(1)}{=} \left(\bigcup_{y\in \mathrm{minf}(X)} y^l \right)^{ul} \stackrel{(2)}{=} (X^l)^{ul} = X^l = \bigwedge_{x\in X} x^l$$

where (1) is given by Proposition 2 and (2) by Proposition 6.

[6] In order to simplify the notation we will write x instead of $\{x\}$.

On the other hand, by Proposition 5 the equality $\bigvee_{x \in X} x^l = X^{ul}$ holds. Then, we have that

$$X^{ul} \stackrel{(1)}{=} \Big(\bigcup_{y \in \mathrm{msup}\, X} y^u \Big)^l \stackrel{(2)}{=} \bigcap_{y \in \mathrm{msup}\, X} y^{ul} \stackrel{(3)}{=} \bigcap_{y \in \mathrm{msup}\, X} y^l \stackrel{(4)}{=} \bigwedge_{y \in \mathrm{msup}\, X} \iota(y)$$

where (1) is given by Proposition 6, the equality (2) holds since $({}^u, {}^l)$ is a Galois connection, (3) because $y^{ul} = y^l$, for all $y \in M$, and (4) is due to Proposition 2. □

5 Dedekind-MacNeille completion and FCA

As stated in the introduction, the Dedekind-MacNeille construction has already played an important role in FCA. As an example, it can be seen as the concept lattice associated to the general ordinal scale associated to a poset, see [7]. Several algorithms for constructing the Dedekind-MacNeille completion of a finite poset have been proposed, for instance, Ganter and Kuznetsov [6] introduced a stepwise method, with cubic complexity, which constructs one new element at a time.

Proposition 7. *Let $(L, \leq)$ be a complete lattice, $(P, \leq)$ be a poset and $\varphi\colon P \to L$ be an order-embedding such that $\varphi(P)$ is both supremum and infimum dense in L. Then $L \cong \mathfrak{B}(P, P, \leq) \cong \mathsf{DM}(P)$.*

Proof. Since $({}^u, {}^l)$ is the Galois connection given by the concept-forming operators associated with the context $(P, P, \leq)$, one easily deduces that $\mathfrak{B}(P, P, \leq) \cong \mathsf{DM}(P)$, since $\mathsf{DM}(P)$ is the set of extensions of the concept lattice $\mathfrak{B}(P, P, \leq)$, see [7, page 48].

On the other hand, due to the fact that $\varphi\colon P \to L$ is an order-embedding, we have that P and $\varphi(P)$ are isomorphic. Moreover, from Theorem 3 we have that $L \cong \mathsf{DM}(\varphi(P))$. As a result, we obtain the following chain of isomorphisms:

$$L \cong \mathsf{DM}(\varphi(P)) \cong \mathsf{DM}(P) \cong \mathfrak{B}(P, P, \leq)$$

□

Our next goal is to prove that the Dedekind-MacNeille completion "distributes" with respect to the construction of the concept lattice associated to a Galois connection.

Let the pair of mappings $\varphi\colon P \to Q$ and $\psi\colon Q \to P$ be a Galois connection between posets. The following result states that it can be extended to the corresponding completions.

Proposition 8 ([15]). *Any Galois connection $\varphi\colon P \to Q$ and $\psi\colon Q \to P$ between posets, can be uniquely extended to a Galois connection between $\mathsf{DM}(P)$ and $\mathsf{DM}(Q)$.*

This extension $(\bar{\varphi}, \bar{\psi})$ is given by

$$\bar{\varphi}(A^{ul}) = \bigwedge_{x \in A} \iota_Q(\varphi(x)), \text{ for all } A \subseteq P \text{ and } \bar{\psi}(B^{ul}) = \bigwedge_{y \in B} \iota_P(\psi(y)), \text{ for all } B \subseteq Q.$$

From now on, in order to simplify the notation, we will erase the subscripts from the mappings ι_P and ι_Q, that is, we will write ι instead of ι_P or ι_Q.

We can now state and prove the main result in this paper:

Theorem 4. *Let $(P, \leq)$, $(Q, \leq)$ be posets and (φ, ψ) be a Galois connection between P and Q, the Dedekind-MacNeille completion of concept poset $\mathsf{CP}(P, Q, \varphi, \psi)$ is isomorphic to the concept lattice $\mathsf{CL}(\mathsf{DM}(P), \mathsf{DM}(Q), \bar{\varphi}, \bar{\psi})$, that is*

$$\mathsf{DM}(\mathsf{CP}(P, Q, \varphi, \psi)) \cong \mathsf{CL}(\mathsf{DM}(P), \mathsf{DM}(Q), \bar{\varphi}, \bar{\psi})$$

Proof. From Theorem 3, it is sufficient to show that $\mathsf{CP}(P, Q, \varphi, \psi)$ can be order-embedded as a supremum and infimum dense subset of $\mathsf{CL}(\mathsf{DM}(P), \mathsf{DM}(Q), \bar{\varphi}, \bar{\psi})$.

Let $(X, Y) \in \mathsf{CL}(\mathsf{DM}(P), \mathsf{DM}(Q), \bar{\varphi}, \bar{\psi})$ be an arbitrary element. By $X \in \mathsf{DM}(P)$ and Proposition 5, we have that $X = X^{ul} = \bigvee_{x \in X} x^l$. Moreover, since the Galois connection $(\bar{\varphi}, \bar{\psi})$ is the extension of (φ, ψ), we obtain

$$(X, Y) = \big(\bar{\psi}(\bar{\varphi}(X)), \bar{\varphi}(X)\big) = \big(\bar{\psi}(\bar{\varphi}(X)), \bigwedge_{x \in X} \iota(\varphi(x))\big)$$

$$\bigvee_{x \in X} \big(\iota(\psi(\varphi(x))), \iota(\varphi(x))\big) = \big(\bar{\psi}(\bar{\varphi}(\bigvee_{x \in X} \iota(\psi(\varphi(x))))), \bigwedge_{x \in X} \iota(\varphi(x))\big)$$

Since both are concepts of $\mathsf{CL}(\mathsf{DM}(P), \mathsf{DM}(Q), \bar{\varphi}, \bar{\psi})$ and they have the same intension, they are the same concept and so $\mathsf{CP}(P, Q, \varphi, \psi)$ is supremum dense in $\mathsf{CL}(\mathsf{DM}(P), \mathsf{DM}(Q), \bar{\varphi}, \bar{\psi})$.

The proof of infimum dense is similarly obtained. □

6 Conclusions

After recalling the basic notions about FCA, multilattices, and the Dedekind-MacNeille completion, we have studied the properties of the DM-completion of a multilattice in terms of the elements of the multilattice. Moreover, we have proved that the effect of interspersing the DM completion wrt the construction of the concepts is, somehow, distributive.

As future work, we will keep studying the algebraic properties of multilattices in relation to the theory of Formal Concept Analysis; in this respect, it might be interesting considering the potential implications of the *soft left-continuity* introduced in [12].

References

1. L. Antoni, S. Krajci, O. Kridlo, B. Macek, and L. Pisková. On heterogeneous formal contexts. *Fuzzy Sets and Systems*, 234:22–33, 2014.

2. R. Bělohlávek. Concept lattices and order in fuzzy logic. *Annals of Pure and Applied Logic*, 128:277–298, 2004.
3. I.P. Cabrera, P. Cordero, G. Gutiérrez, J. Martínez, and M. Ojeda-Aciego. On residuation in multilattices: filters, congruences, and homomorphisms. *Fuzzy Sets and Systems*, 234:1–21, 2014.
4. M. Cornejo, J. Medina, and E. Ramírez. A comparative study of adjoint triples. *Fuzzy Sets and Systems*, 211:1–14, 2012.
5. B. Davey and H. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, second edition, 2002.
6. B. Ganter and S. O. Kuznetsov. Stepwise construction of the Dedekind-MacNeille completion. *Lecture Notes in Computer Science*, 1453:295–302, 1998.
7. B. Ganter and R. Wille. *Formal Concept Analysis: Mathematical Foundation*. Springer Verlag, 1999.
8. G. Georgescu and A. Popescu. Concept lattices and similarity in non-commutative fuzzy logic. *Fundamenta Informaticae*, 53(1):23–54, 2002.
9. H. MacNeille. Partially ordered sets. *Trans. Amer. Math. Soc.*, 42(416–460), 1937.
10. J. Medina, M. Ojeda-Aciego, and J. Ruiz-Calviño. Fuzzy logic programming via multilattices. *Fuzzy Sets and Systems*, 158(6):674–688, 2007.
11. J. Medina, M. Ojeda-Aciego, and J. Ruiz-Calviño. Formal concept analysis via multi-adjoint concept lattices. *Fuzzy Sets and Systems*, 160(2):130–144, 2009.
12. J. Medina, M. Ojeda-Aciego, and J. Ruiz-Calviño. Concept-forming operators on multilattices. In *Intl Conf on Formal Concept Analysis, Lecture Notes in Artificial Intelligence 7880*, pages 203–215, 2013.
13. S. Pollandt. *Fuzzy Begriffe*. Springer, Berlin, 1997.
14. S. Roman. *Lattices and ordered sets*. Springer, 2009.
15. Z. Shmuely. The structure of Galois connections. *Pacific Journal of Mathematics*, 54(2):209–225, 1974.