Constructing right adjoints between fuzzy preordered sets⋆

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Abstract. In this work, we focus on adjunctions (also named isotone Galois connections) between fuzzy preordered sets; specifically, we study necessary and sufficient conditions that have to be fulfilled in order such an adjunction to exist.

Keywords: Galois connection, Adjunction, Preorder, Fuzzy sets

1 Introduction

Adjunctions (also called isotone Galois connection) between two mathematical structures provide a means of linking both theories allowing for mutual cooperative advantages.

A number of results can be found in the literature concerning sufficient or necessary conditions for a Galois connection between ordered structures to exist. The main result of this paper is related to the existence and construction of the right adjoint to a given mapping f, but *in a more general framework*. It is worth to recall that, in a fuzzy setting, reflexivity and antisymmetry are conflicting properties [1] and, whereas some authors [4] opted for dropping reflexivity, our choice in this case has been to ignore antisymmetry and, therefore, consider fuzzy preorders.

Hence, our initial setting is to consider a mapping $f\colon A \to B$ from a fuzzy preordered set A into an unstructured set B, and then characterize those situations in which B can be fuzzy preordered and an isotone mapping $g\colon B \to A$ can be built such that the pair (f, g) is an adjunction.

A set of necessary conditions for an adjunction to exist between fuzzy preordered sets was introduced in [6]. The main contribution in this paper is to prove that the necessary conditions are also sufficient.

The structure of this work is the following: in the next section, we introduce the preliminary definitions and results, essentially notions related to fuzzy preorderings and to Galois connections, and some results which will be later needed. Section 3 introduces several lemmas which allow to simplify the presentation of the proof of the main result in Section 4, where the construction of the right adjoint is given based on the set of necessary conditions already known from [6].

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2 Preliminary definitions and results

As usual, we will consider a *residuated lattice* $\mathbb{L} = (L, \vee, \wedge, \top, \bot, \otimes, \rightarrow)$ as underlying structure for considering the generalization to a fuzzy framework.

An $\mathbb{L}$-*fuzzy set* is a mapping from the universe set, say X, to the lattice L, i.e. $X\colon U \to L$, where $X(u)$ means the degree in which u belongs to X.

Given X and Y two $\mathbb{L}$-fuzzy sets, X is said to *be included in* Y, denoted as $X \subseteq Y$, if $X(u) \le Y(u)$ for all $u \in U$.

An $\mathbb{L}$-*fuzzy binary relation on* U is an $\mathbb{L}$-fuzzy subset of $U \times U$, that is $\rho_U\colon U \times U \to L$, and it is said to be:

- *Reflexive* if $\rho_U(a, a) = \top$ for all $a \in U$.
- *Transitive* if $\rho_U(a, b) \otimes \rho_U(b, c) \le \rho_U(a, c)$ for all $a, b, c \in U$.
- *Symmetric* if $\rho_U(a, b) = \rho_U(b, a)$ for all $a, b \in U$.
- *Antisymmetric* if $\rho_U(a, b) = \rho_U(b, a) = \top$ implies $a = b$, for all $a, b \in U$.

Definition 1 (Fuzzy poset / fuzzy preordered set).
An $\mathbb{L}$-fuzzy poset *is a pair* $\mathbb{U} = (U, \rho_U)$ *in which* ρ_U *is a reflexive, antisymmetric and transitive* $\mathbb{L}$*-fuzzy relation on* U*.*

An $\mathbb{L}$-fuzzy preordered set *is a pair* $\mathbb{U} = (U, \rho_U)$ *in which* ρ_U *is a reflexive and transitive* $\mathbb{L}$*-fuzzy relation on* U*.*

A crisp (pre-)ordering can be given in U *by* $a \le_U b$ *if and only if* $\rho_U(a, b) = \top$*.*

From now on, when no confusion arises, we will omit the prefix "$\mathbb{L}$-".

Definition 2. *For every element* $a \in U$*, the extension to the fuzzy setting of the notions of* upset *and* downset of the element a *are defined by* $a^{\uparrow}, a^{\downarrow}\colon U \to L$ *where* $a^{\downarrow}(u) = \rho_U(u, a)$ *and* $a^{\uparrow}(u) = \rho_U(a, u)$ *for all* $u \in U$*.*

An element $a \in U$ *is an* upper bound *for a fuzzy set* X *if* $X \subseteq a^{\downarrow}$*. The (crisp) set of upper bounds of* X *is denoted by* $UB(X)$*. An element* $a \in U$ *is a* maximum *for a fuzzy set* X *if it is an upper bound and* $X(a) = \top$*.*

The definitions of lower bound *and* minimum *are similar.*

Note that, because of antisymmetry, maximum and minimum elements are necessarily unique.

Definition 3. *Let* $\mathbb{A} = (A, \rho_A)$ *and* $\mathbb{B} = (B, \rho_B)$ *be fuzzy posets.*

1. *A mapping* $f\colon A \to B$ *is said to be* isotone *if* $\rho_A(a_1, a_2) \le \rho_B(f(a_1), f(a_2))$ *for each* $a_1, a_2 \in A$*.*
2. *Moreover, a mapping* $f\colon A \to A$ *is said to be* inflationary *if* $\rho_A(a, f(a)) = \top$ *for all* $a \in A$*. Similarly. a mapping* f *is* deflationary *if* $\rho_A(f(a), a) = \top$ *for all* $a \in A$*.*

Definition 4 (Adjunction). *Let* $\mathbb{A} = (A, \rho_A)$*,* $\mathbb{B} = (B, \rho_B)$ *be fuzzy posets, and two mappings* $f\colon A \to B$ *and* $g\colon B \to A$*. The pair* (f, g) *forms an* adjunction *between* A *and* B*, denoted* $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$ *if, for all* $a \in A$ *and* $b \in B$*, the equality* $\rho_A(a, g(b)) = \rho_B(f(a), b)$ *holds.*

Notation 1 *From now on, we will use the following notation, for a mapping* $f\colon A \to B$ *and a fuzzy subset* Y *of* B*, the fuzzy set* $f^{-1}(Y)$ *is defined as* $f^{-1}(Y)(a) = Y(f(a))$*, for all* $a \in A$*.*

Finally, we recall the following theorem which states different equivalent forms to define an adjunction between fuzzy posets.

Theorem 1 ([5]). *Let* $\mathbb{A} = (A, \rho_A)$*,* $\mathbb{B} = (B, \rho_B)$ *be fuzzy posets, and two mappings* $f\colon A \to B$ *and* $g\colon B \to A$*. The following conditions are equivalent:*

1. $(f, g)\colon \mathbb{A} \leftrightharpoons \mathbb{B}$.
2. *f and g are isotone, $g \circ f$ is inflationary, and $f \circ g$ is deflationary.*
3. $f(a)^{\uparrow} = g^{-1}(a^{\uparrow})$ *for all* $a \in A$*.*
4. $g(b)^{\downarrow} = f^{-1}(b^{\downarrow})$ *for all* $b \in B$*.*
5. *f is isotone and* $g(b) = \max f^{-1}(b^{\downarrow})$ *for all* $b \in B$*.*
6. *g is isotone and* $f(a) = \min g^{-1}(a^{\uparrow})$ *for each* $a \in A$*.*

The next theorem characterizes the situation in which a mapping from a fuzzy poset to an unstructured set has a right adjoint (between fuzzy posets).

Theorem 2 ([7]). *Let* (A, ρ_A) *be a fuzzy poset and a mapping* $f\colon A \longrightarrow B$*. Let* A_f *be the quotient set over the kernel relation* $a \equiv_f b \iff f(a) = f(b)$*. Then, there exists a fuzzy order* ρ_B *in* B *and a map* $g\colon B \longrightarrow A$ *such that* $(f, g)\colon A \leftrightharpoons B$ *if and only if the following conditions hold:*

1. *There exists* $\max[a]_f$ *for all* $a \in A$*.*
2. $\rho_A(a_1, a_2) \le \rho_A(\max[a_1]_f, \max[a_2]_f)$*, for all* $a_1, a_2 \in A$*.*

3 Building adjunctions between fuzzy preordered sets

In this section we start the generalization of Theorem 2 above to the framework of fuzzy preordered sets.

The construction will follow that given in [9] as much as possible. Therefore, we need to define a suitable fuzzy version of the p-kernel relation.

Firstly, we need to set the corresponding fuzzy notion of transitive closure of a fuzzy relation, and this is done via the definition below:

Definition 5 (Transitive closure). *Given a fuzzy relation* $S\colon U \times U \to L$*, for all* $n \in \mathbb{N}$*, the iterations* $S^n\colon U \times U \to L$ *are recursively defined by the base case* $S^1 = S$ *and, then,*

$$S^n(a, b) = \bigvee_{x \in U} \Big(S^{n-1}(a, x) \otimes S(x, b) \Big)$$

The transitive closure *of* S *is a fuzzy relation* $S^{tr}\colon U \times U \to L$ *defined by*

$$S^{tr}(a, b) = \bigvee_{n=1}^{\infty} S^n(a, b)$$

The relation $\approx_A$ allows for gettting rid of the absence of antisymmetry, by linking together elements which are 'almost coincident'; formally, the relation $\approx_A$ is defined on a fuzzy preordered set (A, ρ_A) as follows:

$$(a_1 \approx_A a_2) = \rho_A(a_1, a_2) \otimes \rho_A(a_2, a_1) \qquad \text{for } a_1, a_2 \in A$$

The *kernel equivalence relation* $\equiv_f$ associated to a mapping $f \colon A \to B$ is defined as follows for $a_1, a_2 \in A$:

$$(a_1 \equiv_f a_2) = \begin{cases} \bot & \text{if } f(a_1) \neq f(a_2) \\ \top & \text{if } f(a_1) = f(a_2) \end{cases}$$

Definition 6 (Fuzzy p-kernel). *Let $\mathbb{A} = (A, \rho_A)$ be a fuzzy preordered set, and $f \colon A \to B$ a mapping. The* fuzzy p-kernel *relation $\cong_A$ is the fuzzy equivalence relation obtained as the transitive closure of the union of the relations $\approx_A$ and $\equiv_f$.*

Notice that the fuzzy equivalence classes $[a]_{\cong_A} : A \to L$ are fuzzy sets, whose definition is the following:

$$[a]_{\cong_A}(x) = (x \cong_A a)$$

Lemma 1. *Let $\mathbb{A} = (A, \rho_A)$ be a fuzzy preordered set, and $f \colon A \to B$ a mapping. Then, $a_1 \cong_A a_2 = \top$ if and only if $[a_1]_{\cong_A} = [a_2]_{\cong_A}$.*

Proof. Consider $a_1, a_2 \in A$ such that $a_1 \cong_A a_2 = \top$, and let us prove that $[a_1]_{\cong_A}(u) = [a_2](u)$ for all $u \in A$. Given $u \in A$, by using the neutral element of the product, and symmetry and transitivity of $\cong_A$, we have that

$$(a_1 \cong_A u) = \top \otimes (a_1 \cong_A u) = (a_2 \cong_A a_1) \otimes (a_1 \cong_A u) \leq (a_2 \cong_A u)$$

Similarly, $(a_2 \cong_A u) \leq (a_1 \cong_A u)$ and, therefore, $[a_1]_{\cong_A}(u) = [a_2]_{\cong_A}(u)$ for all $u \in A$. $\square$

All the preliminary notions about fuzzy posets introduced in the previous section carry over fuzzy preordered sets. Note, however, that there is an important difference which justifies the introduction of special terminology concerning maximum or minimum element of a fuzzy subset X: due to the absence of antisymmetry, *there exists a crisp set of maxima* (resp. minima) for X, not necessarily a singleton, which we will denote p-max(X) (resp., p-min(X)).

The following definitions recall the notion of Hoare ordering between crisp subsets, and then we introduce an alternative statement in the subsequent lemma:

Definition 7. *Given a fuzzy preordered set (A, ρ_A), and C, D crisp subsets of A, we define the following relations*

– $(C \sqsubseteq_W D) = \bigvee_{c \in C} \bigvee_{d \in D} \rho_A(c, d)$

– $(C \sqsubseteq_H D) = \bigwedge_{c\in C} \bigvee_{d\in D} \rho_A(c,d)$
– $(C \sqsubseteq_S D) = \bigwedge_{c\in C} \bigwedge_{d\in D} \rho_A(c,d)$

Lemma 2 ([6]). *Consider a fuzzy preordered set* (A,ρ_A), *and* $X, Y \subseteq A$ *such that* $\text{p-min}\, X \neq \varnothing \neq \text{p-min}\, Y$, *then*

$$\big(\text{p-min}\, X \sqsubseteq_W \text{p-min}\, Y\big) = \big(\text{p-min}\, X \sqsubseteq_H \text{p-min}\, Y\big) = \big(\text{p-min}\, X \sqsubseteq_S \text{p-min}\, Y\big)$$

and their value coincides with $\rho_A(x,y)$ *for any* $x \in \text{p-min}\, X$ *and* $y \in \text{p-min}\, Y$.

In [8], given a crisp poset $(A,\leq_A)$ and a map $f\colon A \to B$, it was proved that there exists an ordering $\leq_B$ in B and a map $g\colon B \to A$ such that (f,g) is a crisp adjunction between posets from $(A,\leq_A)$ to $(B,\leq_B)$ *if and only if*

(I) There exists $\max([a]_{\equiv_f})$ for all $a \in A$.
(II) $a_1 \leq_A a_2$ implies $\max([a_1]_{\equiv_f}) \leq_A \max([a_2]_{\equiv_f})$, for all $a_1, a_2 \in A$.

where $\equiv_f$ is the kernel relation associated to f.

These two conditions are closely related to the different characterizations of the notion of adjunction, as stated in Theorem 1 (items 5 and 6); specifically, condition (I) above states that if $b \in B$ and $f(a) = b$, then necessarily $g(b) = \max([a]_{\equiv_f})$, whereas condition (II) is related to the isotonicity of both f and g.

Later, in [9], the previous result was extended to give necessary and sufficient conditions to ensure similar result in the framework of crisp preordered sets. Specifically, it was proved that given any (crisp) preordered set $\mathbb{A} = (A, \lesssim_A)$ and a mapping $f\colon A \to B$, there exists a preorder $\mathbb{B} = (B, \lesssim_B)$ and $g\colon B \to A$ such that (f,g) forms a crisp adjunction between $\mathbb{A}$ and $\mathbb{B}$ *if and only if* there exists a subset S of A such that the following conditions hold:

(i) $S \subseteq \bigcup_{a\in A} \text{p-max}[a]_{\cong_A}$
(ii) $\text{p-min}(UB[a]_{\cong_A} \cap S) \neq \varnothing$, for all $a \in A$.
(iii) If $a_1 \lesssim_A a_2$, then $\Big(\text{p-min}(UB[a_1]_{\cong_A} \cap\ S) \sqsubseteq_H \text{p-min}(UB[a_2]_{\cong_A} \cap\ S)\Big)$, for $a_1, a_2 \in A$.

It is worth to mention that in the conditions above all the notions used are the corresponding crisp versions of those defined in this paper.

In some sense, the conditions (i), (ii), (iii) reflect the considerations given in the previous paragraph, but the different underlying ordered structure leads to a different formalization. Formally, condition (I) above is split into (i) and (ii), since in a preordered setting, if $b \in B$ and $f(a) = b$, then $g(b)$ needs not be in the same class as a but being maximum in its class, as (i) states. However, the latter condition is too weak and (ii) provides exactly the remaining requirements needed in order to adequately reproduce the desired properties for g. Now, condition (iii) is just the rephrasing of (II) in terms of the properties described in (ii).

Finally, in [6], it was proved that the natural extension of the previous conditions to the fuzzy case are also necessary conditions to ensure the existence of an adjunction between fuzzy preordered sets. Specifically,

Theorem 3. *Given fuzzy preordered sets* $\mathbb{A} = (A, \rho_A)$ *and* $\mathbb{B} = (B, \rho_B)$*, and mappings* $f: A \to B$ *and* $g: B \to A$ *such that* $(f, g) : \mathbb{A} \leftrightharpoons \mathbb{B}$ *then*

1. $gf(A) \subseteq \bigcup_{a \in A} \text{p-max}[a]_{\cong_A}$
2. $\text{p-min}(UB[a]_{\cong_A} \cap gf(A)) \neq \varnothing$*, for all* $a \in A$.
3. $\rho_A(a_1, a_2) \leq \Big(\text{p-min}(UB[a_1]_{\cong_A} \cap gf(A)) \sqsubseteq_H \text{p-min}(UB[a_2]_{\cong_A} \cap gf(A))\Big)$ *for all* $a_1, a_2 \in A$.

As a consequence of the previous theorem, a necessary condition for f to be a left adjoint is the existence of a subset $S \subseteq A$ such that the following conditions hold for all $a, a_1, a_2 \in A$:

$$S \subseteq \bigcup_{a \in A} \text{p-max}[a]_{\cong_A} \tag{1}$$

$$\varphi_S(a) \neq \varnothing, \tag{2}$$

$$\rho_A(a_1, a_2) \leq \big(\varphi_S(a_1) \sqsubseteq_H \varphi_S(a_2)\big) \tag{3}$$

where

$$\varphi_S(a) \stackrel{def}{=} \text{p-min}(UB[a]_{\cong_A} \cap S). \tag{4}$$

Remark 1. Notice that, by Lemma 2, $\big(\varphi_S(a_1) \sqsubseteq_H \varphi_S(a_2)\big) = \rho_A(x, y)$ for any $x \in \varphi_S(a_1)$ and $y \in \varphi_S(a_2)$, and this justifies that, in order to simplify the notation, we write $\rho_A(\varphi_S(a_1), \varphi_S(a_2))$ instead of $\big(\varphi_S(a_1) \sqsubseteq_H \varphi_S(a_2)\big)$.

The main contribution in this paper is to show the converse, namely, that the conditions above are also sufficient so that f is a left adjoint.

4 Construction of the right adjoint

In this section, given $f: A \to B$ with the conditions above, we will construct a fuzzy preordering on B together with a mapping $g: B \to A$, which will turn out to be a right adjoint to f.

Definition 8. *Consider a fuzzy preordered set* $\mathbb{A} = (A, \rho_A)$ *together with a mapping* $f: A \to B$ *and a subset* $S \subseteq A$ *satisfying the ambient hypotheses* (1), (2) *and* (3).

For all $a_0 \in A$*, we define the fuzzy relation* $\rho_B^{a_0}: B \times B \to L$ *as follows*

$$\rho_B^{a_0}(b_1, b_2) = \rho_A(\varphi_S(a_1), \varphi_S(a_2))$$

where $a_i \in f^{-1}(b_i)$ *if* $f^{-1}(b_i) \neq \varnothing$ *and* $a_i = a_0$ *otherwise, for each* $i \in \{1, 2\}$.

Notice that the definition might depend largely on the possible choices of a_i; the following lemma, based on Remark 1, shows that the value of $\rho_B^{a_0}$ actually is independent of these choices.

Lemma 3. *The fuzzy relation $\rho_B^{a_0}$ is well-defined, and it is a fuzzy preordering in B.*

Proof. The definition does not depend on the choice of preimages a_i since, if other preimages $\bar{a}_i$ would have been chosen, then $(a_i \equiv_f \bar{a}_i) = \top$ and, hence, by Lemma 1, the fuzzy sets corresponding to the equivalence classes $[a_i]_{\cong_A}$ and $[\bar{a}_i]_{\cong_A}$ would coincide and $\varphi_S(a_i) = \varphi_S(\bar{a}_i)$. Moreover, by Remark 1, we have that

$$\rho_A(\varphi_S(a_1), \varphi_S(a_2)) = \rho_A(x, y) \quad \text{for any } x \in \varphi(a_1) \text{ and } y \in \varphi(a_2)$$

whose value is independent from the choice of x and y.

From the reflexivity of ρ_A, it is straightforward that $\rho_B^{a_0}$ is reflexive. Finally, it is just a matter of easy computations to check that $\rho_B^{a_0}$ is transitive. □

We can now focus on the definition of suitable mappings $g\colon B \to A$ such that (f, g) forms an adjoint pair.

Lemma 4. *Let $\mathbb{A} = (A, \rho_A)$ be a fuzzy preordered set, $f\colon A \to B$ be a mapping and S be a subset of A satisfying the ambient hypotheses* (1), (2) *and* (3). *Given $a_0 \in A$, then there exists a mapping $g\colon B \to A$ such that $(f, g)\colon (A, \rho_A) \leftrightharpoons (B, \rho_B^{a_0})$ where $\rho_B^{a_0}$ is the fuzzy preordering introduced in Definition 8.*

Proof. There is a number of suitable definitions of $g\colon B \to A$, and all of them can be specified as follows:

(C1) If $b \in f(A)$, then $g(b)$ is any element in $\varphi_S(x_b)$ for some $x_b \in f^{-1}(b)$.
(C2) If $b \notin f(A)$, then $g(b)$ is any element in $\varphi_S(a_0)$.

The existence of g is clear by the axiom of choice, since for all $b \in f(A)$, the sets $f^{-1}(b)$ are nonempty (so x_b can be chosen for all $b \in f(A)$) and, moreover, by ambient hypothesis (2), $\varphi_S(x_b)$ and $\varphi_S(a_0)$ are nonempty as well.

Now, we have to prove that g is a right adjoint to f, that is, for all $a \in A$ and $b \in B$ the following equality holds

$$\rho_B^{a_0}\big(f(a), b\big) = \rho_A\big(a, g(b)\big)$$

By definition of $\rho_B^{a_0}$ (see Definition 8), we have that

$$\rho_B^{a_0}(f(a), b) = \rho_A(\varphi_S(a), \varphi_S(w))$$

where w satisfies either $w \in f^{-1}(b)$ if $b \in f(A)$ (therefore, we can choose w to be x_b above) or, otherwise, $w = a_0$. In either case, $g(b) \in \varphi_S(w)$ by construction (namely, (C1) and (C2)). Thus,

$$\rho_B^{a_0}(f(a), b) = \rho_A(x, g(b)) \text{ for any } x \in \varphi_S(a) \tag{5}$$

The proof will be finished if we show that, fixing $x \in \varphi_S(a)$, we can show the equality $\rho_A(x, g(b)) = \rho_A(a, g(b))$.

Firstly, by definition of φ_S, see (4), note that $x \in \varphi_S(a)$ implies $\rho_A(a, x) = \top$ and, hence, we have that

$$\rho_A\big(x, g(b)\big) = \rho_A(a, x) \otimes \rho_A\big(x, g(b)\big) \leq \rho_A\big(a, g(b)\big) \tag{6}$$

For the other inequality, using ambient hypothesis (3), we have

$$\rho_A\big(a, g(b)\big) \leq \rho_A\big(\varphi_S(a), \varphi_S\big(g(b)\big)\big) = \rho_A(x, y) \tag{7}$$

for any $x \in \varphi_S(a)$ and $y \in \varphi_S(g(b))$.

Since $y \in \varphi_S\big(g(b)\big)$ we have that $\rho_A(y, \alpha) = \top$ for all $\alpha \in UB[g(b)]_{\cong_A} \cap S$; on the other hand, since $g(b) \in S$ then $g(b) \in \text{p-max}[g(b)]_{\cong_A}$, particularly $g(b) \in UB[g(b)]_{\cong_A}$, hence $g(b) \in UB[g(b)]_{\cong_A} \cap S$. As a result, we obtain $\rho_A\big(y, g(b)\big) = \top$. Now, connecting expression (7) with transitivity of ρ_A,

$$\rho_A\big(a, g(b)\big) \leq \rho_A(x, y) = \rho_A(x, y) \otimes \rho_A\big(y, g(b)\big) \leq \rho_A\big(x, g(b)\big) \tag{8}$$

for all $x \in \varphi_S(a)$. Joining Equations (6) and (8) we obtain, $\rho_A(x, g(b)) = \rho_A(a, g(b))$ and, finally, Equation (5) leads to

$$\rho_B^{a_0}\big(f(a), b\big) = \rho_A(a, g(b)).$$

□

We can now conclude this section by stating the necessary and sufficient conditions for the existence of right adjoint from a fuzzy preorder to an unstructured set. In this statement, for readability reasons, we do not use the syntactic sugared version of the previous lemma (namely, φ_S) but, instead, state the conditions directly in their low level appearance.

Theorem 4. *Given a fuzzy preordered set $\mathbb{A} = (A, \rho_A)$ together with a mapping $f: A \to B$, there exists a fuzzy preordering ρ_B in B and a mapping $g: B \to A$ such that $(f, g) : \mathbb{A} \leftrightarrows \mathbb{B}$ if and only if there exists $S \subseteq A$ such that, for all $a, a_1, a_2, \in A$:*

1. $S \subseteq \bigcup_{a \in A} \text{p-max}[a]_{\cong_A}$
2. $\text{p-min}(UB[a]_{\cong_A} \cap S) \neq \varnothing$
3. $\rho_A(a_1 a_2) \leq \Big(\text{p-min}(UB[a_1]_{\cong_A} \cap S) \sqsubseteq_H \text{p-min}(UB[a_2]_{\cong_A} \cap S)\Big)$.

Proof. Necessity follows from [6, Thm. 4], considering $S = gf(A)$; sufficiency follows from Lemma 4. □

5 Conclusions

Based on the set of necessary conditions for the existence of right adjunction (between fuzzy preorders) to a mapping $f\colon (A,\rho_A) \to B$, we have proved that these conditions are also sufficient.

It is remarkable the fact that the right adjoint is not unique. In fact, there is a number of degrees of freedom in order to define it: just consider the parameterized construction of g that we have given in terms of an element $a_0 \in A$ (in the case of non-surjective f). Note, however, that our results do not imply that *every* right adjoint should be like that; we simply chose a convenient construction to extent the induced fuzzy ordering on the image of f to the whole set B, and maybe other constructions would be adequate as well (but this is further work).

It is worth to note that there are different versions of antisymmetry and reflexivity in a fuzzy environment (see, for instance, [2, 3]). Accordingly, another line of future work will be the adaptation of the current results to these alternative definitions. Another source of future work will be to study the potential relationship to other approaches based on adequate versions of fuzzy closure systems [10].

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