Logic-based order of magnitude qualitative reasoning for closeness via proximity intervals: a first approach

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Abstract. Qualitative reasoning is an area of AI which provides solutions to problems where the quantitative information either is not available or can not be used; in particular, order of magnitude qualitative reasoning assumes different qualitative classes and relations such as negligibility and closeness. In this paper, we focus mainly on the very important notion of closeness from the logical point of view, which has not received much attention in the literature. Our notion of closeness is based on the so-called *proximity intervals*, which will be used to decide the elements that are close to each other. Some of the intuitions of this definition are explained on the basis of examples. We introduce a multimodal logic for order of magnitude reasoning which includes the notions of closeness and negligibility, we provide an axiom system, which is sound and complete.

1 Introduction

Qualitative reasoning (QR) is very useful for searching solutions to problems about the behavior of physical systems without using exact numerical data. This way, it is possible to reason on incomplete knowledge by providing an abstraction of the numerical values [8, 13] finding solutions to problems that cannot be solved using just a quantitative approach. QR has many applications in AI and, concerning logics for QR, some papers have been focused on Spatio-Temporal Reasoning [2], and about solutions of ordinary differential equations [12]. Moreover, there are recent proposals of logics to deal with movement [10] and qualitative velocities [6].

Another interesting approach to QR is to reason with orders of magnitude [11,14], in which the management of exact values is substituted by reasoning on qualitative classes and relations among them. There are some multimodal logics for order of magnitude reasoning dealing with the relations of negligibility and comparability, see for instance [3, 9]; however, as far as we know, the only published reference on the notion of closeness in a logic-based context is [5], where the notions of closeness and distance are treated using Propositional Dynamic Logic, and their definitions are based on the concept of qualitative sum;

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specifically, in [5] two values are assumed to be close if one of them can be obtained from the other by adding a small number, and small numbers are defined as those belonging to a fixed interval. This specific approach has a number of potential applications but might not be so useful in other situations, for instance, let us consider physical spaces where there are natural or artificial barriers: you can be very close to a place, but if there is a river or a wall, this place is not really so close; we can think of a robot moving in a house for which two points physically close but in different rooms are actually not close. Similarly, one can consider time barriers, such as a deadline to submit an article: if the deadline is, for instance, May 31, the date May 30 can be considered close to the deadline, from the author's point of view, but Jun 1 is not so, because the deadline is already over.

In this paper, we consider a new logic-based alternative to the notion of closeness in the context of multimodal logics. Our notion of closeness stems from the idea that two values are considered to be *close* if they are inside a prescribed area or *proximity interval*. This idea applies to the situations described in the previous paragraph, although it may differ from other intuitions based on distances since it leads to an equivalence relation, particularly, transitivity holds. Neither reflexivity nor symmetry of closeness generate any discussion among the different authors, but transitivity does. The original notion of closeness given by Raiman in [11] allows a certain form of transitivity which he had to tame by using a number of arbitrary limitations to avoid an unrestricted application of chaining. This arbitrariness was criticized in [1], in which a fuzzy set-based approach for handling relative orders of magnitude was introduced. It is remarkable to note that the criticism was made against the arbitrary limitations on chaining the relation, or the impossibility of considering suitable modified versions of transitivity, but not on transitivity per se.

The limitations stated above do not apply to our approach, which can be seen as founded on the notion of granularity as given in [7], which was already suggested in [15]. The main difficulties in accepting closeness as a transitive relation arise in a distance-based interpretation because, then, its unrestricted use would collapse the relation since all the elements would be close. As stated above, our notion will be based not on distance but on membership to a certain element of a given set of proximity intervals, since our driving force is to define an abstract framework for dealing with natural or artificial barriers.

On the other hand, the negligibility notion provided in this paper is a slight generalization of the one given in [4] where, following the line of other classical approaches, for instance [14], the class of 0 is considered to be just a singleton. This choice makes little sense in a qualitative approach, since considering the class of 0 to be just a singleton would require to have measures with infinite precision. Instead, in this paper, we consider the qualitative class INF of *infinitesimals* which, of course, will be all close to each other. Note that these infinitesimals will be interpreted as numbers indistinguishable from 0 in the sense that their difference cannot be measured, not in the sense of hyperreal numbers.

We introduce a multimodal logic for order of magnitude reasoning which manages the notions of closeness and negligibility, then an axiom system is introduced and its soundness and completeness proved. In addition, the decidability of this logic is shown by proving the strong finite model property. The main contributions of the paper are the new definition of closeness and its treatment in a multimodal logic context; it is worth to remark that, although the proofs of completeness and decidability are based on standard techniques (step by step method and filtrations) the specific nature of our logic-based approach makes that the results are not straightforward and, as a result, their proofs are technically interesting.

2 Closeness and negligibility

We will consider a strictly ordered set of real numbers $(\mathbb{S}, <)$ divided into the following qualitative classes:

$$\begin{array}{lll} \text{NL} = (-\infty, -\gamma) & & \text{PS} = (+\alpha, +\beta] \\ \text{NM} = [-\gamma, -\beta) & \text{INF} = [-\alpha, +\alpha] & \text{PM} = (+\beta, +\gamma] \\ \text{NS} = [-\beta, -\alpha) & & \text{PL} = (+\gamma, +\infty) \end{array}$$

Note that all the intervals are considered relative to $\mathbb{S}$.

The labels correspond to “negative large” (NL), “negative medium”(NM), “negative small”(NS), “infinitesimals”(INF), “positive small” (PS), “positive medium” (PM) and “positive large” (PL). It is worth to note that this classification is slightly more general than the standard one [14], since the qualitative class containing the element 0, i.e. INF, needs not be a singleton; this allows for considering values very close to zero as null values in practice, which is more in line with a qualitative approach where accurate measurements are not always possible.

Let us now introduce the notion of closeness. As stated in the introduction, the intuitive idea underlying our notion of closeness is that, in real life problems, there are situations in which we consciously choose not to distinguish between *certain* pairs of elements (for instance, two cars priced 19 000 € and 18 000 € might be both acceptable, but perhaps 20 000 € is considered too expensive for our budget). Somehow, there exist some areas of indistinguishability so that x is said to be close to y if and only if both x and y belong to the same area (although, in the example, 18 000 and 20 000 are equidistant to 19 000, the psychological perception[1] is that 20 000 might be too expensive and, therefore, it is not considered close to 19 000).

We will consider each qualitative class to be divided into disjoint intervals called *proximity intervals*, as shown in Figure 1. The qualitative class INF is itself one proximity interval.

Definition 1. *Let* $(\mathbb{S}, <)$ *be a strictly linear ordering divided into the qualitative classes defined above.*

[1] This is a well-known effect in marketing.

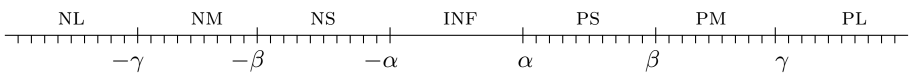

Fig. 1. Proximity intervals.

- *An* r-proximity structure *is a finite set* $\mathcal{I}(\mathbb{S}) = \{I_1, I_2, \ldots, I_r\}$ *of intervals in* $\mathbb{S}$*, such that:*
  1. *For all* $I_i, I_j \in \mathcal{I}(\mathbb{S})$*, if* $i \neq j$*, then* $I_i \cap I_j = \varnothing$*.*
  2. $I_1 \cup I_2 \cup \cdots \cup I_r = \mathbb{S}$*.*
  3. *For all* $x, y \in \mathbb{S}$ *and* $I_i \in \mathcal{I}(\mathbb{S})$*, if* $x, y \in I_i$*, then* x, y *belong to the same qualitative class.*
  4. $\textsc{inf} \in \mathcal{I}(\mathbb{S})$*.*
- *Given a proximity structure* $\mathcal{I}(\mathbb{S})$*, the binary relation of closeness* $\mathfrak{c}$ *is defined, for all* $x, y \in \mathbb{S}$*, as follows:* $x \,\mathfrak{c}\, y$ *if and only if there exists* $I_i \in \mathcal{I}(\mathbb{S})$ *such that* $x, y \in I_i$*.*

Notice that, by definition, the number of proximity intervals is finite, regardless of the cardinality of the set $\mathbb{S}$. This choice is justified by the nature of the measuring devices that after reaching a certain limit, they do not distinguish among nearly equal amounts; for instance, consider the limits to represent numbers in a pocket calculator, thermometer, speedometer, etc.

As a result of considering just finitely many proximity intervals, it can be the case that two elements exist whose magnitudes are not comparable but, according to this approach, turn out to be comparable. In everyday life, we often face similar situations where excessively large quantities are no longer considered to have an appreciable difference.

For instance, if the limit of users simultaneously connected to a server is, say, 1 000 000 users, it is clear that the response would be the same than if 10 000 000 users are connected to the server. In this case, although these quantities may not be comparable in absolute terms, they turn out to be comparable from the point of view of the response of the server. Nevertheless, if for some reason, we need these quantities to be not comparable, we have just to change the choice of the qualitative classes in our approach.

From now on, we will denote by $\mathcal{Q} = \{\textsc{nl}, \textsc{nm}, \textsc{ns}, \textsc{inf}, \textsc{ps}, \textsc{pm}, \textsc{pl}\}$ the set of qualitative classes, and by QC to any element of $\mathcal{Q}$.

The following proposition is an immediate consequence of the definition.

Proposition 1. *The relation* $\mathfrak{c}$ *defined above has the following properties:*

1. $\mathfrak{c}$ *is an equivalence relation on* $\mathbb{S}$*.*
2. *For all* $x, y, z \in \mathbb{S}$*, the following holds:*
   (a) *If* $x, y \in \textsc{inf}$*, then* $x \,\mathfrak{c}\, y$*.*
   (b) *For every* $\textsc{qc} \in \mathcal{Q}$*, if* $x \in \textsc{qc}$ *and* $x \,\mathfrak{c}\, y$*, then* $y \in \textsc{qc}$*.*

The informal notion of negligibility we will use in this paper is the following: x is said to be *negligible* with respect to y if and only if either (i) x is infinitesimal and y is not, or (ii) x is small (but not infinitesimal) and y is *sufficiently large*.

Definition 2. *Let* $(\mathbb{S}, <)$ *be a strictly linear divided into the qualitative classes defined above. The binary relation of negligibility* $\mathfrak{n}$ *is defined on* $\mathbb{S}$ *as* $x \,\mathfrak{n}\, y$ *if and only if one of the following situations holds:*

(i) $x \in \textsc{inf}$ *and* $y \notin \textsc{inf}$*,*
(ii) $x \in \textsc{ns} \cup \textsc{ps}$ *and* $y \in \textsc{nl} \cup \textsc{pl}$*.*

The following straightforward result states some interesting properties about the interaction between the relations of closeness and negligibility.

Proposition 2. *For all* $x, y, z \in \mathbb{S}$ *we have:*

(i) *If* $x \,\mathfrak{c}\, y$ *and* $y \,\mathfrak{n}\, z$*, then* $x \,\mathfrak{n}\, z$*.*
(ii) *If* $x \,\mathfrak{n}\, y$ *and* $y \,\mathfrak{c}\, z$*, then* $x \,\mathfrak{n}\, z$*.*

In order to further explain the underlying behavior of the definitions of closeness and negligibility, we include the following example.

Example 1. It does not rain very often in a city like Málaga (Spain), but there is a real danger of floods due to torrential rain. For this reason, there is a reservoir very close to the city in order to control the water flooding from the mountains. We represent the quantity of water in the reservoir by positive qualitative classes $\textsc{ps}, \textsc{pm}, \textsc{pl}$, and consider the proximity intervals, with a length that can be considered as non-significative with respect to the total quantity of water of the reservoir, for instance 1 000 litres. Using the previous definitions, we can express many interesting situations. For instance, assume that $\textsc{ps}$ is the level of water in the reservoir considered OK (that is, safe), with $\textsc{pm}$ a warning message must be shown, and $\textsc{pl}$ is a dangerous situation that forces to open the floodgates. If the quantity of water x is OK, that is $x \in \textsc{ps}$ and some rain is expected such as the quantity of water will increase to y, several results are possible out of which we detail the following: (a) if $x \,\mathfrak{c}\, y$, then $y \in \textsc{ps}$, meaning that if a small rain is expected, then the situation will remain OK and there is no need to open the floodgates; (b) if $x \,\mathfrak{n}\, y$, then $y \in \textsc{pl}$, which means that if a big rain is expected, the floodgates must be opened and some water has to be released.

The properties given in Proposition 2 can be used in this context. If a small rain is coming ($x \,\mathfrak{c}\, y$) an after that a big rain is expected ($y \,\mathfrak{n}\, z$), then the floodgates must be opened ($x \,\mathfrak{n}\, z$), by using property (i). Similarly, property (ii) in Proposition 2 can be used whenever a big rain is coming ($x \,\mathfrak{n}\, y$) followed by a small rain ($y \,\mathfrak{c}\, z$), so the floodgates must be opened also ($x \,\mathfrak{n}\, z$).

3 Syntax and semantics of $\mathcal{L}(MQ)^{\mathcal{P}}$

In this section, we will use as special modal connectives $\overrightarrow{\Box}$ and $\overleftarrow{\Box}$ to deal with the usual ordering $<$, so $\overrightarrow{\Box}A$ and $\overleftarrow{\Box}A$ have the informal readings: *A is true for* *all numbers greater than the current one* and A *is true for all number less than the current one*, respectively. Two other modal operators will be used, $\boxed{\mathsf{c}}$ for closeness, where the informal reading of $\boxed{\mathsf{c}}A$ is: A *is true for all number close to the current one*, and $\boxed{\mathsf{n}}$ for negligibility, where $\boxed{\mathsf{n}}A$ means A *is true for all number with respect to the current one is negligible*.

The alphabet of the language $\mathcal{L}(MQ)^{\mathcal{P}}$ is defined by using a stock of atoms or propositional variables, $\mathcal{V}$, the classical connectives $\neg, \wedge, \vee$ and $\rightarrow$; the constants for milestones $\alpha^-, \alpha^+, \beta^-, \beta^+, \gamma^-, \gamma^+$; a finite set $\mathcal{C}$ of constants for proximity intervals, $\mathcal{C} = \{c_1, \ldots, c_r\}$ [2]; the unary modal connectives $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, $\boxed{\mathsf{n}}$, $\boxed{\mathsf{c}}$, and the parentheses '(' and ')'. We define the formulas of $\mathcal{L}(MQ)^{\mathcal{P}}$ as follows:

$$A = p \mid \xi \mid c_i \mid \neg A \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid \overrightarrow{\Box}A \mid \overleftarrow{\Box}A \mid \boxed{\mathsf{n}}A \mid \boxed{\mathsf{c}}A$$

where $p \in \mathcal{V}$, $\xi \in \{\alpha^+, \alpha^-, \beta^+, \beta^-, \gamma^+, \gamma^-\}$ and $c_i \in \mathcal{C}$. In order to refer to any constant for positive milestones as α^+ we will use ξ^+ and for negative ones as β^- we will use ξ^-.

The *mirror image* of a formula A is the result of replacing in A each occurrence of $\overrightarrow{\Box}$, $\overleftarrow{\Box}$, α^+, β^+ and γ^+ respectively by $\overleftarrow{\Box}$, $\overrightarrow{\Box}$, α^-, β^- and γ^- and reciprocally. We will use the symbols $\overrightarrow{\Diamond}, \overleftarrow{\Diamond}, \langle\mathsf{c}\rangle, \langle\mathsf{n}\rangle$ as abbreviations, respectively, of $\neg\overrightarrow{\Box}\neg$, $\neg\overleftarrow{\Box}\neg$, $\neg\boxed{\mathsf{c}}\neg$ and $\neg\boxed{\mathsf{n}}\neg$. Moreover, we will introduce $\mathsf{nl}, \ldots \mathsf{pl}$ as abbreviations for qualitative classes, for instance, ps for $(\overleftarrow{\Diamond}\alpha^+ \wedge \overrightarrow{\Diamond}\beta^+) \vee \beta^+$. By means of qc we denote any element of the set $\{\mathsf{nl}, \mathsf{nm}, \mathsf{ns}, \mathsf{inf}, \mathsf{ps}, \mathsf{pm}, \mathsf{pl}\}$.

The cardinality r of the set $\mathcal{C}$ of constants for proximity intervals will play an important role since it, somehow, encodes the granularity of the underlying logic. This implies that, actually, *we are introducing a family of logics which depend parametrically on* r.

Definition 3. *A* multimodal qualitative frame *for* $\mathcal{L}(MQ)^{\mathcal{P}}$ *(a* frame*, for short) is a tuple* $\Sigma = (\mathbb{S}, \mathcal{D}, <, \mathcal{I}(\mathbb{S}), \mathcal{P})$*, where:*

1. $(\mathbb{S}, <)$ *is a strict linearly ordered set.*
2. $\mathcal{D} = \{+\alpha, -\alpha, +\beta, -\beta, +\gamma, -\gamma\}$ *is a set of designated points in* $\mathbb{S}$ *(called* milestones*).*
3. $\mathcal{I}(\mathbb{S})$ *is an r-proximity structure.*
4. $\mathcal{P}$ *is a bijection (called* proximity function*),* $\mathcal{P}\colon \mathcal{C} \longrightarrow \mathcal{I}(\mathbb{S})$*, that assigns to each proximity constant c a proximity interval.*

Definition 4. *Let* Σ *be a frame for* $\mathcal{L}(MQ)^{\mathcal{P}}$*, a* multimodal qualitative model on Σ *(a MQ-*model*, for short) is an ordered pair* $\mathcal{M} = (\Sigma, h)$*, where h is a* meaning function *(or,* interpretation*)* $h\colon \mathcal{V} \longrightarrow 2^{\mathbb{S}}$*. Any interpretation can be uniquely extended to the set of all formulas in* $\mathcal{L}(MQ)^{\mathcal{P}}$ *(also denoted by h) by means of the usual conditions for the classical Boolean connectives and the*

[2] There are at least as many elements in $\mathcal{C}$ as qualitative classes.

following conditions:

$$h(\overrightarrow{\Box}A) = \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } x < y\}$$
$$h(\overleftarrow{\Box}A) = \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } y < x\}$$
$$h(\boxed{\scriptstyle c}A) = \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } x \,\mathfrak{c}\, y\}$$
$$h(\boxed{\scriptstyle n}A) = \{x \in \mathbb{S} \mid y \in h(A) \textit{ for all } y \textit{ such that } x \,\mathfrak{n}\, y\}$$
$$h(\alpha^+) = \{+\alpha\} \qquad h(\beta^+) = \{+\beta\} \qquad h(\gamma^+) = \{+\gamma\}$$
$$h(\alpha^-) = \{-\alpha\} \qquad h(\beta^-) = \{-\beta\} \qquad h(\gamma^-) = \{-\gamma\}$$
$$h(c_i) = \{x \in \mathbb{S} \mid x \in \mathcal{P}(c_i)\}$$

The definitions of *truth*, *satisfiability* and *validity* are the usual ones.

4 An axiom system for $\mathcal{L}(MQ)^{\mathcal{P}}$

In this section we consider the axiom system $MQ^{\mathcal{P}}$ for the language $\mathcal{L}(MQ)^{\mathcal{P}}$, consisting of all the tautologies of classical propositional logic together with the following axiom schemata and rules of inference:

For white connectives

K1 $\overrightarrow{\Box}(A \to B) \to (\overrightarrow{\Box}A \to \overrightarrow{\Box}B)$
K2 $A \to \overrightarrow{\Box}\overleftarrow{\Diamond}A$
K3 $\overrightarrow{\Box}A \to \overrightarrow{\Box}\overrightarrow{\Box}A$
K4 $\big(\overrightarrow{\Box}(A \vee B) \wedge \overrightarrow{\Box}(\overrightarrow{\Box}A \vee B) \wedge \overrightarrow{\Box}(A \vee \overrightarrow{\Box}B)\big) \to \big(\overrightarrow{\Box}A \vee \overrightarrow{\Box}B\big)$

For constants $\xi \in \{\alpha^+, \beta^+, \gamma^+, \alpha^-, \beta^-, \gamma^-\}$

c1 $\overleftarrow{\Diamond}\xi \vee \xi \vee \overrightarrow{\Diamond}\xi$
c2 $\xi \to (\overleftarrow{\Box}\neg\xi \wedge \overrightarrow{\Box}\neg\xi)$
c3 $\gamma^- \to \overrightarrow{\Diamond}\beta^-$
c4 $\beta^- \to \overrightarrow{\Diamond}\alpha^-$
c5 $\alpha^- \to \overrightarrow{\Diamond}\alpha^+$
c6 $\alpha^+ \to \overrightarrow{\Diamond}\beta^+$
c7 $\beta^+ \to \overrightarrow{\Diamond}\gamma^+$

For proximity constants (for all $i, j \in \{1, \ldots, n\}$)

p1 $\bigvee_{i=1}^{n} c_i$
p2 $c_i \to \neg c_j$ (for $i \neq j$)
p3 $(\overleftarrow{\Diamond}c_i \wedge \overrightarrow{\Diamond}c_i) \to c_i$
p4 $\overleftarrow{\Diamond}c_i \vee c_i \vee \overrightarrow{\Diamond}c_i$

Mixed axioms (for all $i \in \{1, \ldots, n\}$)

m1 $(c_i \wedge \mathsf{qc}) \to \big(\overleftarrow{\Box}(c_i \to \mathsf{qc}) \wedge \overrightarrow{\Box}(c_i \to \mathsf{qc})\big)$
m2 $(c_i \wedge \mathsf{inf}) \to \big(\overleftarrow{\Box}(\mathsf{inf} \to c_i) \wedge \overrightarrow{\Box}(\mathsf{inf} \to c_i)\big)$
m3 $\boxed{\scriptstyle c}A \leftrightarrow \Big(A \wedge \bigvee_{i=1}^{r}\Big(c_i \wedge \overleftarrow{\Box}(c_i \to A) \wedge \overrightarrow{\Box}(c_i \to A)\Big)\Big)$

m4 $\boxdot A \leftrightarrow \Big(\Big(\mathsf{inf} \to (\overleftarrow{\Box}(\neg\mathsf{inf} \to A) \wedge \overrightarrow{\Box}(\neg\mathsf{inf} \to A))\Big) \wedge$
$$\Big((\mathsf{ns} \vee \mathsf{ps}) \to (\overleftarrow{\Box}(\mathsf{nl} \to A) \wedge \overrightarrow{\Box}(\mathsf{pl} \to A))\Big)\Big)$$

The mirror images of **K1, K2** and **K4** are also considered as axioms.

Rules of inference:

(MP) Modus Ponens for $\to$.
(N$\overrightarrow{\Box}$) If $\vdash A$ then $\vdash \overrightarrow{\Box} A$.
(N$\overleftarrow{\Box}$) If $\vdash A$ then $\vdash \overleftarrow{\Box} A$.

The syntactical notions of *theorem* and *proof* for $MQ^{\mathcal{P}}$ are defined as usual.

Example 2. The aim of this example is to specify in $\mathcal{L}(MQ)^{\mathcal{P}}$ the behavior of a device to automatically control the speed of a car. Assume the system has, ideally, to maintain the speed close to some speed limit v. For practical purposes, any value in an interval $[v-\varepsilon, v+\varepsilon]$ for small ε is admissible. The extreme points of this interval can then be considered as the milestones $-\alpha$ and $+\alpha$ of our frames; on the other hand, we will consider different levels of velocity in a qualitative approach ranging from *very slow* to *very fast*. We will introduce consequently the atoms $\mathtt{v}_{-3}, \mathtt{v}_{-2}, \mathtt{v}_{-1}, \mathtt{v}_0, \mathtt{v}_1, \mathtt{v}_2, \mathtt{v}_3$ associated to them (which are interpreted, respectively, as the qualitative classes NL, NM, NS, INF, PS, PM, PL and, moreover, $\mathtt{v}_0$ represents the interval $[v-\varepsilon, v+\varepsilon]$).

We will introduce also the atoms `accelerate`, `maintain`, `release` and `brake` to describe actions of the system with their intuitive meaning.

Now we represent how the system works:

1. Whenever the speed is below the intended limit, then the engine is accelerated, whereas when the speed is within the admissible limits, the speed is maintained. Thus, we have the two formulas below
$$(\mathtt{v}_{-3} \vee \mathtt{v}_{-2} \vee \mathtt{v}_{-1}) \to \mathtt{accelerate} \qquad \mathtt{v}_0 \to \mathtt{maintain}$$
2. It can happen that the speed increases more than the limit allowed due to external factors, for instance when the road has negative slope, this way some rules are required to maintain the speed. Usually, when the car reaches the speed limit, the driver does not brake immediately but releases the accelerator instead, so that the air friction helps to recover an admissible speed. We accomplish this action precisely in the proximity interval immediately after $\mathtt{v}_0$, which we will call, say, c. As a result, we have the two formulas
$$c \to (\overleftarrow{\Diamond}\alpha^+ \wedge \overleftarrow{\Box}(\overleftarrow{\Diamond}\alpha^+ \to c)) \qquad c \to \mathtt{release}$$
3. When we are beyond the limit imposed by the interval c, then the system has to *actively* brake:
$$(\neg c \wedge \overleftarrow{\Diamond} c) \to \mathtt{brake}$$

According to the intended meaning of the previous formulas, the atoms `accelerate`, `keep`, `release` and `brake` are true, respectively, at

$$\text{NL} \cup \text{NM} \cup \text{NS} \qquad \text{INF} \qquad I_c \qquad (\text{PS} \setminus I_c) \cup \text{PM} \cup \text{PL}$$

where I_c is the proximity interval represented by c. Note that the length of this interval depends on the granularity of the system; indeed, given the axiom **m1**, I_c should be included in the class PS.

Some consequences of the behavior of the system (specifically, valid formulas in the model) are the following:

$\mathtt{brake} \to \overrightarrow{\Box}\mathtt{brake}$
(If the system brakes at a specific speed, then it brakes at higher speeds)

$\mathtt{release} \to \boxed{c}(\mathtt{v}_1 \wedge \neg\mathtt{brake})$
(If the throttle is released at certain speed, then any small variation implies that the speed is still slightly fast and the system does not brake)

$\mathtt{release} \to \boxed{n}(\mathtt{v}_{-3} \to \mathtt{accelerate})$
(If the throttle is released at certain speed and, by any circumstances, the speed decreases excessively, then it has to accelerate again)

$\mathtt{accelerate} \to \boxed{c}\neg\mathtt{brake}$
(If the system accelerates, it will not brake immediately)

$\mathtt{v}_0 \to \overrightarrow{\Box}(\mathtt{v}_2 \to \overleftarrow{\Diamond}\mathtt{release})$
(The throttle is released before reaching a fast speed)

The space limit does not allow to include details on the expressivity of the axiom system introduced above; but it can be proved to be sound and complete for the semantics established at the beginning of the paper and, furthermore, decidible.

5 Conclusions and future work

Logics for order of magnitude reasoning are important to deal with situations where numerical values are either imprecise or unavailable. Negligibility and closeness are two important relations in the area of qualitative reasoning and, in this paper, we have presented a sound, complete and decidable multimodal logic for order of magnitude reasoning which considers negligibility and a new approach to closeness, based on *proximity intervals*, a notion which turned out to be a useful tool to decide whether two elements are close to each other.

As future work, we are studying the computational complexity of the satisfiability problem in this logic.

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