Consistency of qualitative judgements from discrete t-conorms

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Abstract. The contribution deals with discrete smooth t–conorms, and their application to solve the problem of the consistency of qualitative judgements in preference structures. An algorithm to compute this consistency is also included.

Keywords: Preference structures, cardinal representation, smooth discrete t–conorm, decision making, qualitative judgement.

1 Introduction

A preference structure on a set of alternatives A is a triplet (P, I, J) of binary relations on A defining a decision maker's preferences, as follows: aPb if and only if the decision maker prefers a to b; aIb if and only if the decision maker is indifferent between alternatives a and b, and aJb if and only the decision maker is unable to compare a and b. These structures are well-studied mathematical structures in the theory of preference modelling [11].

In decision making under uncertainty, the construction of appropriate models to represent the preferences of decision makers is needed. Decision models are traditionally quantitative but, recently, qualitative models have been developed in order to construct models closer to the natural language of decision maker opinions (see [2–4]). In this way, our approach is based on providing preferential information about two alternatives at a time, firstly by giving a judgement as to their relative attractiveness (ordinal judgement) and secondly, if the two alternatives are not deemed to be equally attractive, by expressing a qualitative judgement about the difference of attractiveness between the most attractive of the two alternatives and the other. In [7] six semantic categories of difference of attractiveness are used: "very weak", "weak", "moderate", "strong", "very strong" and "extreme". Fundamental references are [10, 5] where applications of Measurement Theory to problems of Decision Making can be found.

This contribution is organized as follows. Section 2 is devoted to introduce basic definitions and results used along the paper. Section 3 contains the main results of the paper. They deal with the use of smooth t–conorms to perform the aggregation of elementary judgements. Last Section includes an algorithm to compute the consistency of any preference linear inequalities system.

2 Preliminaries

Despite the fact that t-norms were first introduced in the context of statistical metric spaces, they have become an important tool in many other fields: fuzzy sets, decision making, statistics, theories of non-additive measures, etc. Comprehensive monographs on t-norms are [1, 6]. According to the fact that in most practical situations it is necessary to discretize the real unit interval, we need to deal with logics where the set of truth values is modelled by a finite linearly ordered set $L = \{0, 1, \ldots, n\}$. As it is expected, in the definition of discrete triangular norms (discrete t–norms, for short) we use the set of axioms provided by Schweizer and Sklar [12] once adapted to this finite setting. Thus, our requirements on a t–norm $T : L \times L \to L$ for all a, b, c, d in L are:

(i) $T(a, b) = T(b, a)$,
(ii) $T(T(a, b), c) = T(a, T(b, c))$,
(iii) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$,
(iv) $T(a, n) = a$.

The following are the three basic discrete t–norms T_M, T_L and T_D:

- $T_M(a, b) = \min(a, b)$, (minimum)
- $T_L(a, b) = \max(a + b - n, 0)$, (Łukasiewicz t–norm)
- $T_D(a, b) = \begin{cases} \min(a, b), & \text{if } a = n \text{ or } b = n \\ 0, & \text{otherwise} \end{cases}$ (drastic t–norm).

Note that discrete triangular conorms (t–conorms) can be introduced as dual functions of discrete t–norms: $S(a, b) = n - T(n - a, n - b)$. Thus the axioms for a t–conorm are:

(i) $S(a, b) = S(b, a)$,
(ii) $S(S(a, b), c) = S(a, S(b, c))$,
(iii) $S(a, b) \leq S(c, d)$ whenever $a \leq c$ and $b \leq d$,
(iv) $S(a, 0) = a$.

The dual of the three basic t–norms are the following t–conorms:

- $S_M(a, b) = \max(a, b)$, (maximum)
- $S_L(a, b) = \min(a + b, n)$, (Łukasiewicz t–conorm or bounded sum)
- $S_D(a, b) = \begin{cases} \max(a, b), & \text{if } a = 0 \text{ or } b = 0 \\ n, & \text{otherwise} \end{cases}$ (drastic t–conorm).

As said before, in this paper we use t–conorms to model the aggregation of elementary preference values taken in an ordinal scale $L = \{0, 1, \ldots, n\}$. Thus, for our purpose, we describe below only those concepts and results for discrete t–conorms that we need in this paper.

Two fundamental classes of discrete t–conorms have been considered: the class of smooth discrete t–conorms ($S(a + 1, b) - S(a, b) \leq 1$) and the class of Archimedean discrete t–conorms ($S(a, a) > a$ for all $a \neq 0, n$). In [8] it is proved that the only smooth and Archimedean discrete t–conorm is the Łukasiewicz t–conorm $S_L(a,b) = \min(a+b,n)$, and then a characterization of the class of smooth discrete t–conorms is given: S is smooth if and only if there exists a subset I of L, $I = \{0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r < \alpha_{r+1} = n\}$ such that S has the following structure:

$$S(x,y) = \begin{cases} \min(\alpha_{i+1}, x+y-\alpha_i), & \text{if } (x,y) \in (\alpha_i, \alpha_{i+1})^2,\ 0 \le i \le r \\ \max(x,y), & \text{otherwise} \end{cases}$$

That is, a discrete t–conorm is smooth if and only if it is an ordinal sum of smooth Archimedean discrete t–conorms. Taking into account that smoothness is the proper equivalent of the continuity of ordinary t–conorms, the given representation of smooth discrete t–conorms is in full analogy to the Aczél-Ling's representation of ordinary continuous t–conorms. As in the case of ordinary t–conorms, no representation theorems exist so far for the class of all discrete t–conorms. More details on discrete t-norms and t–conorms can be found in [8, 9].

A binary relation P defined on a finite set A is called a strict weak order[1] if there exists a complete pre-order R on A such that xPy if and only if xRy and yR^cx.

From a strict weak order P, we can obtain a linearly ordered partition of A by considering the indifference relation associated with P: xIy if and only if xP^cy, and yP^cx. The quotient set A/I can be linearly ordered in the following way: $\overline{x} > \overline{y}$ if and only if xPy. We denote this ordered partition, sometimes called a ranking, by $(A_1, \ldots, A_p)$.

3 Qualitative judgements based on smooth t-conorms

Let A be a finite set with cardinality $m \geq 2$ and $L = \{0, 1, \ldots, n\}$, $n \geq 1$, a finite linearly ordered set. Given a binary relation P on A and $\delta : P \to \{1, \ldots, n\}$, we consider the L-valued binary relation $\Delta : A \times A \to L$ defined by

$$\Delta(x,y) = \begin{cases} 0, & \text{if } xP^cy \\ \delta(x,y), & \text{if } xPy \end{cases} .$$

Definition 1 *We say that $\mu : A \to \mathbb{R}$ is a (cardinal) weak representation of (A, P, Δ) if the following conditions hold:*

1. $xPy \Leftrightarrow \mu(x) > \mu(y)$,
2. $\Delta(x,y) > \Delta(z,t) \Rightarrow \mu(x) - \mu(y) > \mu(z) - \mu(t)$. [2]

[1] A strict weak order is a mathematical formalization of the intuitive notion of a ranking of a set, some of whose members may be tied with each other. A weak order is simply a complete pre-order. A reflexive binary relation is often called a weak preference, and from a weak preference R we can define xPy if and only if xRy and yR^cx. We say that P is a strict preference (derived from R).

[2] Note that the axiom 2 only requires the implication in one direction.

Remark 1 *Let $(A_1, \ldots, A_p)$, $1 \leq p \leq n$, be the linearly ordered partition of A given by a strict weak order P. Then we have from condition 1 in Definition 1 that for all $i = 1, \ldots, p$,*

$$x, y \in A_i \Longleftrightarrow \mu(x) = \mu(y)$$

Thus it is sufficient to define μ on one element of each A_i. We will write $\mu(A_i)$ to indicate $\mu(x)$ for any $x \in A_i$. Moreover we will use the notation $\delta_{i,j} = \Delta(A_i, A_j) = \Delta(x, y)$, with $x \in A_i, y \in A_j$.

Observe also that μ is completely determined by giving its value on A_1 and the differences $\mu(A_i) - \mu(A_i + 1)$, $i = 1, \ldots, p-1$. We will indicate these differences by $x_i = \mu(A_i) - \mu(A_i + 1)$.

With all this notation, the condition 2 in Definition 1 becomes

$$\delta_{i,j} > \delta_{k,l} \Longrightarrow x_i + \cdots + x_{j-1} > x_k + \cdots + x_{l-1} \tag{1}$$

The set of all inequalities obtained from (1) will be called a preference inequalities system.

Proposition 1 *If $\mu : A \to \mathbb{R}$ is a weak representation of (A, P, Δ) then:*

(i) *P is a strict weak order,*
(ii) *If xIx', yIy' and xPy then we have $x'Py'$ and $\Delta(x, y) = \Delta(x', y')$,*
(iii) *$\delta_{ij} \geq \max(\delta_{ik}, \delta_{kj})$ for all $1 \leq i < k < j \leq p$.*

A straightforward way of giving the set of $\delta_{i,j},\ i < j$, is to start with a given list of elementary values $\delta_{r,r+1}$, and then combining them by means of an appropriate function F, as follows: $\delta_{i,j} = F(\delta_{i,i+1}, \ldots, \delta_{j-1,j})$.

It is worth to point out that conditions (i) to (iii) in Proposition 1 are not sufficient to ensure that there exists a weak representation of (A, P, Δ), as the following example shows.

Example 1 *Let $L = \{0, 1, 2, 3, 4\}$, and a ranking $(A_1, \ldots, A_5)$. Consider a function $F : \bigcup_{r=1}^{4} L^r \to L$ such that $F \geq \max$. If we take $\delta_{1,2} = \delta_{2,3} = 1$, $\delta_{3,4} = 3$, $\delta_{4,5} = 4$, and $\delta_{1,3} = F(1, 1) = 6$, $\delta_{3,5} = F(3, 4) = 4$, and then the corresponding preference inequalities system is given by*

$$\begin{aligned}
&\delta_{3,4} > \delta_{1,2} \Longrightarrow x_3 > x_1 \\
&\delta_{3,4} > \delta_{2,3} \Longrightarrow x_3 > x_2 \\
&\delta_{4,5} > \delta_{1,2} \Longrightarrow x_4 > x_1 \\
&\delta_{4,5} > \delta_{2,3} \Longrightarrow x_4 > x_2 \\
&\delta_{1,3} > \delta_{3,5} \Longrightarrow x_1 + x_2 > x_3 + x_4.
\end{aligned}$$

This system is inconsistent, because from the first and the fourth inequalities or from the second and the third inequalities we obtain that $x_3 + x_4 > x_1 + x_2$ which is in contradiction with the last inequality of the system.

A natural way of obtaining the preferences $\delta_{i,j}$ is to combine the elementary judgements $\delta_{k,k+1}$. Having in mind the condition (iii) in Proposition 1 we propose the use of t–conorms to do that. We think that the use of this kind of aggregation functions is appropriate because their properties are rather adequate for our purpose.

Proposition 2 *Let (A, P, Δ) be such that:*

(i) P is a strict weak order,
(ii) If xIx', yIy' and xPy then we have $x'Py'$ and $\Delta(x, y) = \Delta(x', y')$,
(iii) $\delta_{ij} = S(\delta_{i,i+1}, \ldots, \delta_{j-1,j})$ for all $i < j$, where S is a smooth t–conorm on L.

Then there exists a weak representation $\mu : A \to \mathbb{R}$ of the triplet (A, P, Δ).

Let us give now a detailed example of this proposition, which has also been computed according to Algorithm 1 given in Section 4.

Example 2 *Let us consider $n = 6$ and $p = 9$. Let us consider the smooth t-conorm S given by the following table:*

6	6	6	6	6	6	6	6
5	5	5	5	5	6	6	6
4	4	4	4	4	5	6	6
3	3	3	3	3	4	5	6
2	2	2	2	3	4	5	6
1	1	1	2	3	4	5	6
0	0	1	2	3	4	5	6
	0	1	2	3	4	5	6

Let us suppose that the list of elementary judgements $\delta_{r,r+1}$ is $4, 5, 1, 1, 4, 5, 1, 2$. Now we calculate the values $\delta_{ij} = S(\delta_{i,i+1}, \ldots, \delta_{j-1,j})$, $i < j$, and represent them by means of the following matrix, in whose diagonal we put the values $\delta_{r,r+1}$:

$$\begin{pmatrix} 4 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 0 & 5 & 5 & 5 & 6 & 6 & 6 & 6 \\ 0 & 0 & 1 & 1 & 4 & 6 & 6 & 6 \\ 0 & 0 & 0 & 1 & 4 & 6 & 6 & 6 \\ 0 & 0 & 0 & 0 & 4 & 6 & 6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 5 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We will use a new collection of variables $y_1, \ldots, y_n$, where

$$y_h = x_r \iff \delta_{r,r+1} = h$$

Thus we have to determine $y_1, \ldots, y_n$ such that the condition (1) holds. Since the first non-idempotent element is $m+1 = 4$, we define $y_1 = 1, y_2 = 9, y_3 = 81$, *and then we rescale them by dividing by* $(m+1)y_3 = 4 \cdot 81$, *obtaining* $y_1 = 1/324, y_2 = 1/36, y_3 = 1/4$. *Then the corresponding* x_i *are:*

$\delta_{34} = \delta_{45} = \delta_{89} = 1 \Longrightarrow x_3 = x_4 = x_8 = y_1 = 1/324$

and

$\delta_{78} = 2 \Longrightarrow x_7 = y_2 = 1/36.$

Now, from $m+1=4$ *to the next idempotent element,* $m+s=6$, *we define* $y_{m+a} = a$, *that is,* $y_4 = 1, y_5 = 2$, *and* $y_6 = 3$, *which corresponds to*

$\delta_{12} = \delta_{56} = 4 \Longrightarrow x_1 = x_5 = y_4 = 1$

and

$\delta_{23} = \delta_{67} = 5 \Longrightarrow x_2 = x_6 = y_5 = 2.$

Thus we have taken $x_1 = 1, x_2 = 2, x_3 = 1/324, x_4 = 1/324, x_5 = 1, x_6 = 2, x_7 = 1/36, x_8 = 1/324$.

Let us finally prove that the condition (1) holds. If we observe the matrix above, we obtain the inequalities to be satisfied, but we reduce the list to the inequalities that are relevant.

a) $\delta_{13} = \delta_{57} = 6 > \delta_{24} = \delta_{68} = 5 \Longrightarrow x_1 + x_2, x_5 + x_6 > x_2 + x_3, x_6 + x_7$ *that is,* $x_1 > x_3,\ x_1 + x_2 > x_6 + x_7,\ x_5 + x_6 > x_2 + x_3, x_5 > x_7$ *which are satisfied.*
b) $\delta_{24} = \delta_{68} = 5 > \delta_{35} = 4 \Longrightarrow x_2 + x_3, x_6 + x_7 > x_3 + x_4$ *that is,* $x_2 > x_4,\ x_6 + x_7 > x_3 + x_4$ *which are satisfied.*
c) $\delta_{56} = 4 > \delta_{79} = 2 \Longrightarrow x_5 > x_7 + x_8$ *which is satisfied.*
d) $\delta_{78} = 2 > \delta_{35} = \delta_{89} = 1 \Longrightarrow x_7 > x_3 + x_4, x_8$ *which are satisfied.*

Remark 2 *Note that if* $\mu : A \to \mathbb{R}$ *is a weak representation of* (A, P, Δ), *then* $\mu^{'} = \sigma \circ \mu$, *where* $\sigma : \mathbb{R} \to \mathbb{R}$ *is of the form* $\sigma(t) = at + b$, $a > 0$, *is also a weak representation of that pair. If* μ *and* $\mu^{'}$ *satisfy 1) and 2) in Definition 1, then there exists a strictly increasing function* $\sigma : \mathbb{R} \to \mathbb{R}$ *such that* $\mu^{'} = \sigma \circ \mu$.

In the following example we show that it is not necessary to use a smooth t–conorm in order to ensure the consistency of the preference inequalities system.

Example 3 *Let* S *be the drastic t–conorm on* L. *Let us see that* $y_i = n + i$, $i = 1, \ldots, n$, *and* $x_k = y_i$ *when* $\delta_{k,k+1} = i$, *is a solution of the system of inequalities.*

There are only two possibilities for the inequality $\delta_{ij} > \delta_{kl}$:

- $n = \delta_{ij} > \delta_{k,k+1}$, *with* $j > i+1$, *in which case we have*

$$x_i + \cdots + x_{j-1} \geqslant 2y_1 = 2n + 2 > 2n = y_n > x_k,$$

- $\delta_{i,i+1} > \delta_{k,k+1}$, *and the corresponding inequality* $x_i > x_k$ *is satisfied by the given solution.*

The following example shows the relevance of the associativity of the aggregation function. It has been computed by means of the Algorithm 1.

Example 4 *Consider the following non–associative function F :*

5	5	5	5	5	5	5
4	4	4	4	4	4	5
3	3	3	3	4	4	5
2	2	3	3	3	4	5
1	1	1	3	3	4	5
0	0	1	2	3	4	5
	0	1	2	3	4	5

and let us take the elementary preference degrees $[2 \quad 3 \quad 1 \quad 1 \quad 2 \quad 3 \quad 1 \quad 1 \quad 2]$. *Then the associated matrix of preference degrees is :*

$$\begin{pmatrix} 2&3&3&3&3&4&4&4&4 \\ 0&3&3&3&3&4&4&4&4 \\ 0&0&1&1&3&4&4&4&4 \\ 0&0&0&1&3&4&4&4&4 \\ 0&0&0&0&2&3&3&3&3 \\ 0&0&0&0&0&3&3&3&3 \\ 0&0&0&0&0&0&1&1&3 \\ 0&0&0&0&0&0&0&1&3 \\ 0&0&0&0&0&0&0&0&2 \end{pmatrix}$$

Now we have $\delta_{3,7} = 4 > \delta_{5,10} = 3$*, i.e.,* $x_3+x_4+x_5+x_6 > x_5+x_6+x_7+x_8+x_9$*, that is,* $x_3 + x_4 > x_7 + x_8 + x_9$*, however* $\delta_{3,5} = 1 < \delta_{7,10} = 3$*, i.e.,* $x_3 + x_4 < x_7 + x_8 + x_9$*.*

Therefore, the preference inequalities system is not consistent.

4 Algorithm: Consistency of preference inequalities system

In order to be able to deal with large preference inequalities system an algorithm has been implemented in Matlab to compute the consistency of the system. Its code can be found in Algorithm 1.

5 Conclusions

Some results concerning the problem of consistency of qualitative judgements about the difference of attractiveness between alternatives are obtained. The use of smooth t–conorms on a finite ordinal scale to model the aggregation of elementary qualitative judgements is introduced.

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Algorithm 1 Compute consistency of preference inequalities system $Ax < 0$ s.t. $x > 0$.

```
procedure CONSISTENCY OF PREFERENCE INEQUALITIES SYSTEM FROM A GIVEN AGGREGATION FUNCTION
    Input: A function F of choice (it may be a t–conorm), a vector d of n degrees of preference δ_{i,i+1}.
    for i = 1 to n do MP(i,i) = d(i)
        for j = 1 to n do
            if i < j then u(i) = F(d(i)+1,d(i+1)+1)
                if (i+1) ¡ j then
                    for k = (i+1):(j-1) do u(k) = F(u(k-1)+1,d(k+1)+1)
                    end forMP(i,j) = u(j-1)
                elseMP(i,j) = u(i)
                end if
            elseif i > j MP(i,j) = 0
            end if
        end for
    end for
    EP ← (-1)Identity(n+1); v ← 0; w_1 ← 0; w_2 ← 0;
    MP2(i,j+1) ← MP(i,j); MP2(i,1) ← 0; MP(n+1,j) ← 0;
    Diagonal i-th: D(i) ← (D_{l,i+l})_{l=1,...,n+1-i}
    for i = 1 to n do
        for p = 1 to n do
            for l = 1 to length(D(i)) do
                for k = 1 to length(D(p)) do
                    if D_{l,i+l} > D_{k,p+k} then
                        for j = l to i+l do w_1(j) = −1
                        end for
                        for m = l to p+l do w_2(m) = 1
                        end forv ← w_1 + w_2 EP ← [EP; v] w_1 ← 0 w_2 ← 0
                    end if
                end for
            end for
        end for
    end for
    NewEC ← unique(EC)
    Algorithm of Simplex to NewEC
    if exitflag = 1 then
        write 'The system Ax < 0 s.t. x > 0 with the given function F is CONSISTENT'
    else
        write 'The system Ax < 0 s.t. x > 0 with the given function F is INCONSISTENT'
    end if
end procedure
```